Fostering Basic Problem-Solving Skills in Mathematics*

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*The research described in this paper was supported in part by a grant from the EXXON Education Foundation.

Teaching students to think mathematically has become a national priority. Included in the list of capabilities that would distinguish the mathematically literate from the illiterate is the ability to apply mathematics to the solution of "real-world" problems. In NCTM's new curriculum recommendations for K-12, the first standard for instruction in each of the grade levels reads, "Mathematics as Problem Solving." This emphasis on problem solving is justified with the claim that "mathematical problem solving, in its broadest sense, is nearly synonymous with doing mathematics" [NCTM, 1989, p. 137].

Every fourth year since 1973 the academic performance of a nation-wide sample of students at ages 9, 13, and 17 has been evaluated by NAEP (National Assessment of Educational Progress). The mathematical capabilities across the last three assessments were recently summarized by Dossey, Mullis, Lindquist, and Chambers [1988]. Their summary suggests that, while over the past eight years there have been moderate gains in performance on simple mathematical manipulations, performance on problems requiring multiple steps in the solution process has remained at a disturbingly low level. For example, in the 1986 assessment only 6% of the 17-year-olds and .5% of the 13-year-olds could solve problems similar to the one below:

> Suppose you have 10 coins and have at least one each of a quarter, a dime, a nickel, and a penny. What is the *least* amount of money you could have? (Choices offered: .41, .47*, .50, .82)

The corresponding percentages in the 1978 assessment were 7% and 1%. It would appear that to this point we have made little progress in developing the general reasoning skills that are required to solve such problems.

Problem solving has received considerable attention during the past 15 years among mathematics educators, psychologists and educational researchers [see Kilpatrick, 1985, for an historical overview]. One outcome of these investigations is a general vocabulary of problem solving steps and methods that should prove helpful in the design and evaluation of curriculum that would satisfy the NCTM stamdards. Schoenfeld [1985] has given the most detailed analysis to date of problem-solving skills and heuristics. Some of the more complex heuristics include: examining special cases, constructing an analogous problem, trying a simpler problem, replacing problem conditions by equivalent ones, introducing auxiliary elements, assuming a solution exists and determining its properties, and considering arguments via contradiction, symmetry, or scaling. His analysis provides a technical basis for the hope that mathematics students can be taught to use these heuristics and thereby improve their ability to solve complex problems.

However, our experience over the past eight years teaching remedial mathematics courses at the college-freshman level suggests that approaches that teach these heuristics may already assume more problem-solving facility than many students possess. In this article we first describe what we consider to be the most basic problem-solving skills, skills which are prerequisite to many of those described in the traditional problem-solving literature, and which many students lack. We then present a protocol of a student solving what would appear to be a simple problem and discuss some of the difficulties that prevent her from arriving at a correct solution. Finally we discuss one method of developing these basic skills in the context of a problem-based mathematics course. Although our immediate experience is restricted to college freshman who have difficulties in mathematics, we expect that much of what we have observed is not unique to students in our courses and applies to the teaching of mathematics at the secondary level.

Description of basic problem-solving skills

Chisko and Davis [1986], in their description of a workshop designed to foster problem solving, characterize their instructional approach as one that:

> encourages students to become actively involved with a problem by asking questions such as these: What do we know? What do we want to know? What intermediate information would be useful? What is a reasonable range for a solution? [p. 594]

These questions relate to skills listed in Table 1 under the heading "stage-specific skills." These skills are divided among three major stages that correspond roughly to categories developed by Polya [1945]: a) comprehending and representing, b) planning, assembling, and implementing a solution, c) verifying the solution.

We have added a second category in Table 1 labeled "general skills and attitudes." The skills listed in this category are those that appear to be essential in the successful execution of the three stages in category I. In describing their students' approach to problems at the end of the problem-solving workshop, Chisko and Davis

I. STAGE-SPECIFIC SKILLS

A. Comprehending and Representing

1. Viewing representation as a solution step
2. Finding the goal and the givens
3. Drawing and modifying diagrams

B. Planning, Assembling and Implementing a Solution

1. Breaking the problem into parts (setting sub-goals)
2. Organizing chains of operations or inferences in multi-step problems

C. Verifying the Solution

1. Viewing verification as a solution step
2. Assessing the reasonableness of the answer in terms of initial estimates

II. GENERAL SKILLS AND ATTITUDES

A. Alternately Generating and Evaluating Ideas (as opposed to recalling algorithms)

B. Striving for Precision in the Use of:

1. Inferences
2. Verbal expressions
3. Symbols and diagrams
4. Algorithms

C. Monitoring Progress

1. Making written records to keep track of and organize solution elements and partial results
2. Using confusion as a signal to rethink part of the solution
3. Proceeding slowly in the expectation of making and needing to correct errors

Some Basic Problem-Solving Skills
Table 1

mention several characteristics that exemplify these general skills [p. 595]. We have indicated in brackets below those skills listed in Table 1 to which we think various parts of their description apply:

> They began by reading and rereading carefully [II-C3] defining the problem [I-A2] and discussing each condition. They made appropriate use of charts and drawings to organize the information [I-A3; II-C1]. Throughout, they exhibited confidence that they could solve the problem even though the solution was not readily apparent [II-A].

It is important to stress the fact that the skills listed in Table 1 are not considered to be a complete set of skills necessary in problem solving, but are a subset of those skills that we regard as basic. What is remarkable is that despite the apparent simplicity of the skills listed, many students have not developed these capabilities by the time they are in college. We believe that in teaching problem solving it is fruitful to distinguish these basic skills from the more complex skills and heuristics listed earlier. Below, we illustrate some of these basic skills primarily by noting their absence in a videotaped dialogue between two students who were solving the "Days problem" [adapted from Whimbey, 1977]:

> What day precedes the day after tomorrow if four days ago was two days after Wednesday?

Many of our students find problems like the Days problem challenging and make many errors. This may seem surprising since such problems involve *no mathematics*. All students are intimately familiar with the factual content and subject matter of the problem (the temporal relationships between days of the week) and use this information daily. The problem is simple enough to make the number of

003	S1:	(Reads) "What day precedes the day after tomorrow, if four days ago was two days after Wednesday?" Oh God, this is heavy.
006	S1:	Four days ago was two days after Wednesday and two days after Wednesday is Friday. So, what day precedes the day after tomorrow? So, we back up four days ——
007	S2:	Four days from what?
008	S1:	Friday. So that's —— Monday? —— Alright, so that'd be Monday. Monday was four days ago before Friday . . . I'm so confused . . . Oh God, I don't believe this. "What day precedes the day after tomorrow, if four days ago was two days after Wednesday?"
012	S2:	What was two days after Wednesday?
013	S1:	Friday: —— Okay, I'm going the wrong way. So, what day —— I can't do this.
014	S2:	Sure you can.
019	S1:	"What day precedes" —— it means come after, the day after tomorrow. Okay, four days ago was Friday, so Saturday, then Sunday, then Monday, Tuesday. Tuesday is four days after Friday, so —— so Wednesday precedes —— Wednesday precedes Tuesday, right . . .?
026	S1:	Four days ago was Friday, so that brings you to Tuesday . . . so the day that precedes the day after tomorrow —— so maybe it's Thursday I'm —— I really don't know. ——— 'Cause ——— the day that precedes the day —— alright tomorrow —— huh . . .
033	S1:	What day precedes the day after tomorrow? Okay, okay, that's right, it's Tuesday. That's four days ago was Friday . . . and what day precedes the day after tomorrow. So, it'd be Thursday.
037	S1:	Yeah, 'cause you're at Tuesday, and tomorrow is Wednesday, and the day that precedes the day after Wednesday is Thursday.
043	S1:	Ok so, Thursday —— is the day that precedes the day after tomorrow. So, Thursday.

Note: Numbers running down the left of the table indicate the placement of the statements in the complete transcript. Each dash (—) corresponds to a pause of 2 seconds.

Protocol of a Student's Solution to the Days Problem
Table 2

errors surprising. It was students' persistent difficulties with elementary problems of this sort that prompted an analysis of the fundamental skills required for problem solving.

The two students whose transcript is presented in Table 2 were freshmen in our remedial course.

They were working together on this problem early in the semester using an assigned procedure called "pair problem solving" [Whimbey & Lochhead, 1982]. The first student (S1) was acting as the problem solver with the task of talking aloud while she solved the problem. She eventually arrived at Thursday as the answer to the problem rather than the correct answer, Wednesday. The second student (S2) was acting as the "listener" with the task of encouraging the first student to verbalize her thoughts continuously, and asking for justification of steps in the solution. (The Appendix contains a handout given to students to help them learn the role of the listener and to distinguish it from the more natural role of "helper.")

Analysis of protocol

The dialogue provided in Table 2 has been condensed to approximately one third of the original protocol. The entire protocol gives the impression of even more confusion and difficulty: S1 rereads the entire problem and in effect restarts her solution a total of eight times. In one respect she demonstrates a problem-solving skill in that she spends considerable time simply trying to comprehend the problem. She is apparently aware that the problem is sufficiently complex that she needs to *proceed slowly.* Many students forge ahead with little awareness, arriving at an incorrect answer much more quickly than this student did. In another respect, however, this student's need to repeatedly restart her solution indicates her difficulty in *keeping track of partial results* (such as "today is Tuesday"). The simple use of *written records* as a memory aid eludes her during most of her solution. Nor does it occur to her to make an ordered list or calendar diagram of the days on which she could count backward or forward from a given day. One source of this difficulty is probably the fact that students are given few, if any, multi-step problems in school that require this type of record keeping.

The Days problem is most easily solved by starting with the last phrase in the problem statement. "Two days after Wednesday" is Friday. Given that Friday was "four days ago," counting ahead four days gets us to today — Tuesday. The "day after tomorrow" would then be Thursday, and the "day preceding" that would be Wednesday. The order of the chain of inferences used in this solution is precisely the reverse of the order of the corresponding information in the problem statement. Although S1 eventually proceeds in the correct order through this inference chain, many students fail to do so. Rather, they persist in trying to solve the problem by treating the information in the order that it appears in the problem statement.

In lines 6 and 8 it appears that S1 is counting days in the wrong direction from Friday. She seems to count backwards in time four days to reach Monday instead of counting forward to Tuesday. Perhaps the word "ago" triggers the idea of counting backward. Rather than dismissing this as a freak error, we regard it as a difficulty with *precision in inference* which, in this case, leads to an inference inversion. Two reasons for taking this error seriously are that (a) we have observed many students making similar errors on problems of this sort and (b) S1 was a motivated and fairly conscientious student and was putting considerable effort into this problem. Her later correction (line 19) shows that S1 was capable of correctly computing "4 days ago." We believe that usually students make this error because they underestimate the degree of care and precision required in making such inferences. Students may be accustomed to written assignments in other courses which allow for vague language and less precise inferences. Such vagueness is fatal in mathematics.

In line 19 S1 shows a misunderstanding of the meaning of the word "precedes," which to her means "comes after." This apparently is a simple problem with vocabulary as opposed to a difficulty with problem solving or reasoning. Accepting her unorthodox use of the word "precedes," we find a reasoning error in line 26, where she says, "that brings you to Tuesday . . . so the day that precedes the day after tomorrow — so maybe it's Thursday . . . ." She repeats this argument in lines 33 and 37. Given her assumption about the meaning of "precedes," her answer should be Friday instead of Thursday. The sub-problem she appears to be working on at this point concerns a set of nested relationships: "What is the day after the day after tomorrow, if today is Tuesday?" She seems to drop one of the "after" relations implied by this question to arrive at Thursday instead of Friday. We interpret this as a difficulty in dealing with nested relationships, a type of inference in which an operator is applied to itself. One reason for the difficulty manifested here may be that, while these relationships are seldom used in natural language, they are ubiquitous in mathematics.

In addition to some obvious weaknesses, this student shows some other strengths in problem solving. In line 13 she seems to realize that she has counted in the wrong direction, and in line 19 she counts up four days in the correct direction. Although it is not clear how she detected her error, she *responds positively to her confusion,* evaluating and correcting her first attempt. From our observations this "reading" of confusion is a critical skill in problem solving. Among the ways that expert problem solvers use feelings of uncertainty and more pronounced feelings of confusion are to tag partial results for later scrutiny and to signal that they may not yet have a coherent understanding of the problem. Novices often ignore feelings of confusion with the rationale that since they always feel confusion when working on problems, such feelings ought to be ignored. When the confusion becomes too strong to ignore, they then take it as a sign that no progress can be made and abandon work on the problem. So whereas experts look for *problem specific* causes of feelings of confusion, trying to reduce them by making adjustments to their solution process, novices give these feelings a more general interpretation, ignoring them if possible, or surrendering to them if not. The necessary process of engaging in a cycle of conjecture, evaluation, and correction requires both the attribution of confusion to problem-specific causes and breaking away from the common belief that problem solving always consists of recalling a well-defined procedure and executing it. Rather, problem solving often requires the *alternate generation and evaluation* of methods of representation and solution. The willingness shown by S1 to attack the problem even though she has no clear algorithm for solving it is a partial strength. It is partial because she appears at one point nearly to give up on the problem due to her confusion. The encouragement of her partner (line 14) may have helped her to persist.

In summary, students can have difficulty with problems requiring rudimentary skills. That these include problems that involve virtually no mathematics underscores the point that these are skills that need to be learned *in addition* to mathematical content. Because it seems unlikely that instruction in mathematical operations and concepts will necessarily develop these skills, techniques specifically aimed at fostering them need to be used.

Methods of developing basic skills

A consensus seems to be developing among many mathematics educators that small-group "cooperative work" is a more fruitful structure for promoting problem-solving skills than individual "seat work" [e.g., Easley & Easley, 1982; Lesh, 1981; Schoenfeld, 1985; Whimbey & Lochhead, 1982]. We believe that students can develop these basic skills if they are made aware, and frequently reminded, of the importance of these skills while solving multi-step problems in small groups. As one way to foster this awareness, we have provided students in our remedial-level mathematics courses with the list of prompts in Table 3 and instructed them on how to use it during problem solving. The prompts are clustered in a manner that matches the organization of skills in Table 1.

These prompts are used with a carefully sequenced set of problems, starting with a selection from Whimbey and Lochhead [1982]. As described in that book, having students work in pairs alternating between the roles of solver and listener is used as a means of promoting these basic skills. These conversations between partners promote student awareness of their own conceptualizations and reasoning processes. One of our primary objectives is that students internalize the roles of solver/listener and learn to carry on an internal and critical dialogue with themselves in solving problems. This promotes the constructive cycle of generating an idea, criticizing it, and improving it (II-A in Table 1). This dialectic cycle contrasts with the all-or-nothing strategy of retrieving a suitable algorithm. For many students, abandoning the all-or-nothing strategy requires a drastic change in their attitude towards mathematical problem solving.

Pair problem solving also promotes several other basic skills listed in Table 1. In particular, having to verbalize to a partner encourages what we have called precision. The listener frequently does not understand what it is the solver is saying, and the simple probes, "What do you mean?" or "Can you explain your diagram?" have the effect of sharpening the solver's distinctions and definitions. Frequently, in response to a question posed by the listener, the solver will return to the problem statement, in the process discovering that they have misunderstood the question or omitted important information. Also, diagrams that were originally intended to help the solver communicate to the listener are often used subsequently as a tool for solving the problem.

I DON'T UNDERSTAND THE PROBLEM

Read the problem again.
What do I know; what do I need to know?
What am I trying to find out?
Can I rephrase the problem in my own words?
Can I draw a (better) diagram?

I DON'T KNOW WHERE TO GO FROM HERE

Have I been given relevant information that I haven't used yet?
Can I solve part of the problem?
Is there some useful information hidden in the problem?

IS MY SOLUTION CORRECT?

How confident am I about the solution?
What would a reasonable answer be?
Is each step in my solution valid?
Is there another method I can use to check my answer?

I'M CONFUSED

Be patient. Take it easy and go slowly.
Organize what I have in a better way.

Basic Prompts for Problem Solving
Table 3

Noddings [1985] believes that among the basic skills that are fostered by small groups are organizing pictorial representations and focusing on relevant problem dimensions. We have observed that the explicit roles in pair problem solving are particularly helpful in promoting these same skills. It is natural for the listener to ask the solver to explain the reasons for some action. These types of questions encourage more explicit descriptions of subgoals and strategies for achieving those goals. As students become more articulate about these, they are better able to plan and monitor their problem-solving processes. Having to verbally describe even very simple methods and mathematical ideas can be challenging, and gives the learning activities an integrity that is missing in many classrooms.

Early in the course it is typical for students solving problems together to ask the instructor if their answer to the problem is correct. This question gives the instructor the opportunity to emphasize that verifying the answer is one of the responsibilities of the problem solver. Accordingly, student inquiries about the correctness of their solutions are typically met with the suggestion that they evaluate their confidence in the solution and, if possible, verify that it is correct. The initial frustration expressed by many students to these requests indicates that they have not previously regarded verification as a part of the problem-solving process in educational settings. Indeed, this is almost always a responsibility assumed by the instructor.

Perhaps the most powerful instructional aspect of pair problem solving, however, is the fact that it gives the instructor access to thinking processes and problem-solving strategies that otherwise remain invisible. The instructor, circulating among students as they talk aloud, not only gains information about an individual's strengths and weaknesses, but can also intervene in ways that address problem-solving deficiencies.

To be successful, pair problem solving must be implemented using a sequence of multi-step problems that are appropriate to the students' ability level. Problems cannot be so easy that they require little thought. On the other hand, beginning with problems that are far beyond the capabilities of the students is a guaranteed path to frustration and failure for both students and instructors. According to the NAEP results, the majority of high-school students lack many of the basic skills required to solve multi-step problems. This being the case, the problems used early in a course stressing problem solving ought to focus initially on these rudimentary skills. It would be unfortunate if innovative courses designed to teach problem solving failed because they emphasized the more advance heuristics when students had not yet developed the more basic skills.

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Appendix

LISTENER RESPONSIBILITIES

Responsibility	Examples
1 — Listen carefully	Can you repeat that? Slow down. I'm not following you.
2 — Encourage vocalization	What are you thinking? Can you explain what you're writing?
3 — Ask for clarification	What do you mean? Can you say more about that?
4 — Check for accuracy	Are you sure about that? That doesn't seem right to me.

DO NOT give hints
DO NOT solve the problem yourself
DO NOT tell the solver *how* to correct an error