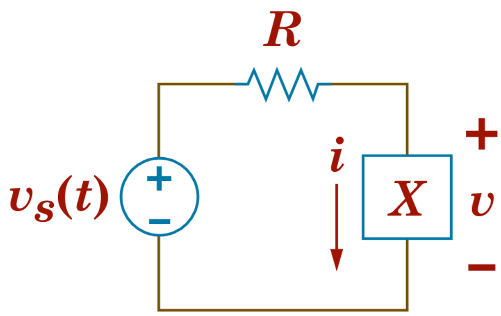
Solving Problems from Module I2

This document is intended to help people learn the rudiments of solving problems in module I2 (Solving RL, RC Networks). You will still need to practice within a variety of contexts, but if you are stuck, perhaps there is something here that will help get you unstuck.

The fundamental network is shown below.



If you can solve this network, you can solve any network, because a Thevenin analysis is always possible; that is, you can turn any network into this one. Element *X* can be an inductor or a capacitor.

The solution to any differential equation can be written:

$$v(t) = v_f(t) + v_n(t)$$

where the subscript *f* refers to the "forced" response, and the subscript *n* refers to the "natural" response. (In a Differential Equations class, these would be called the "particular" and the "homogeneous" solutions.)

If the source is dc, we can show that the forced response is a constant. And when there is just one energy-storage device *X*, the differential equation is "first order", which means the natural response is an exponential function:

$$v_n(t) = Ae^{-t/\tau}$$

where *A* is an unspecified constant, and τ is the "time constant" (*RC* for *X*=*C* and *L*/*R* for *X*=*L*). To find the value of *A*, we need to know the value of the function at some time (other than infinity). The most convenient time is *t*=0+, because then the exponential function is equal to 1, so:

$$v(0^+) = v_f + A$$

or $A = v_{0+} - v_f$.

The general solution is therefore:

$$v(t) = v_f + (v_{0+} - v_f)e^{-t/\tau}$$

for the voltage across *X*. The same form is equally valid for current through *X*:

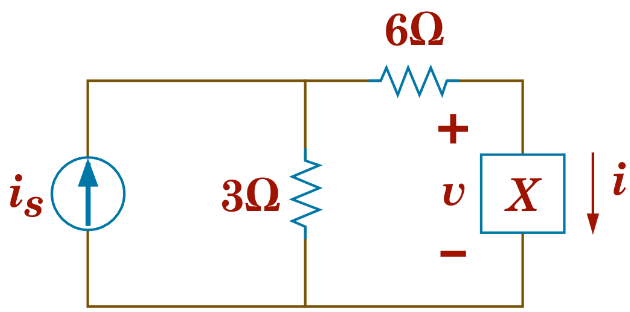
$$i(t) = i_f + (i_{0+} - i_f)e^{-t/\tau}$$

Thus, in order to find these functions of time, we need only find the appropriate time constant and the initial and final values of voltage and current.

Although this will always work, it can be tricky at times to apply to an actual problem. In particular, for an inductor, the voltage changes instantly from zero just before *t*=0 (i.e., $t = 0^-$) to a non-zero value just afterwards ($t = 0^+$). Likewise, the capacitor current changes from zero at $t = 0^-$ to a non-zero value at $t = 0^+$. Since it is not always easy to find the new values, there is a second method that is often recommended.

Capacitor voltage and inductor current are special, because they do not change at *t*=0. (If you look at the voltage-current relationships for *C* and *L*, you can see that if these changed you would need an infinite current through *C* and an infinite voltage across *L*.) Therefore, the initial value of capacitor voltage is the value it had just before *t*=0, which is a steady-state value. Similarly, the initial value of the inductor current is the same as its value just before *t*=0. And since the network is in equilibrium just before *t*=0, these initial values are relatively easy to determine. Then, once you have capacitor voltage or inductor current, you can use their voltage-current relationships to find the other (capacitor current or inductor voltage). An example should help.

Consider the following network:



with:

$$i_s(t) = 12u(t) - 15u(-t)$$

in A. In other words, this is the most general form for a dc source. Note that the value of the current through the source is –15A for all time before t=0 ($t \le 0^-$), and then 12A for all time after t=0 ($t \ge 0^+$).

First, let $X = C = 2\mu\text{F}$. The Thevenin resistance is 9Ω, so the time constant is 18μs. We focus on voltage, because it does not change at t=0 when the source changes. Just before the source changes, the network is in equilibrium, which means no current flows through the capacitor, and it behaves like an open circuit. All the current from the source flows through the 3Ω resistor. The voltage v across the capacitor is the voltage across the 3Ω resistor, or:

$$v_{0-} = -15 \cdot 3 = -45\text{V}$$

Because the capacitor voltage cannot change instantly from one value to another, this is also the voltage just after the source changes:

$$v_{0+} = v_{0-} = -45\text{V}$$

The forced response is also a steady-state (equilibrium) response, since the source has now been on with the same value of 12A for a very long time. Again, at steady-state, all the current from the source flows through the 3Ω resistor, and the voltage across the capacitor is:

$$v_f = 12 \cdot 3 = 36\text{V}$$

Therefore:

$$v(t) = 36 - 81e^{-t/18\mu}$$

The current through the capacitor is now found using its v-i relation, as shown here:

$$\begin{aligned} i(t) &= C \cdot dv/dt \\ &= 2\mu \cdot \left(-1/18\mu \cdot -81e^{-t/18\mu}\right) \\ &= 9e^{-t/72\mu} \end{aligned}$$

Therefore, the initial current (just after t=0) is 9A, i.e., the coefficient of the exponential.

Next, let $X = L = 36\,\text{mH}$. The Thevenin resistance is still 9Ω, so the time constant is 4ms. This time, we focus on current, because it does not change when the source changes. Before t=0, the network is in equilibrium, so the voltage across L is zero, which means it behaves like a wire. The current from the source splits, with 2/3 flowing through 3Ω and 1/3 flowing through 6Ω. Therefore:

$$i_{0-} = -15 \cdot 1/3 = -5\text{A}$$

Because the inductor current cannot change instantly from one value to another, this is also the current just after the source changes:

$$i_{0+} = i_{0-} = -5\text{A}$$

The forced response is also a steady-state response, since the source has now been on for a very long time after t=0, so:

$$i_f = 12 \cdot 1/3 = 4\text{A}$$

and the solution is:

$$i(t) = 4 - 9e^{-t/4\text{m}}$$

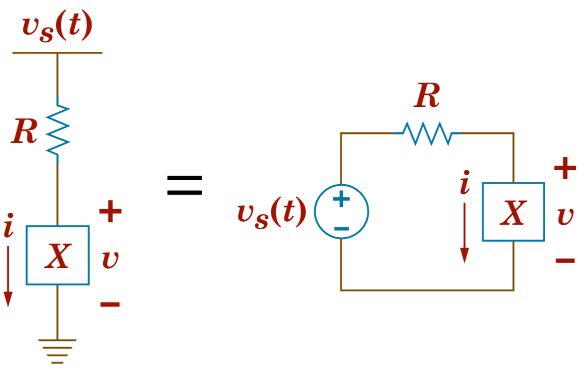
Then:

$$\begin{aligned} v(t) &= L \cdot di/dt \\ &= 36\text{m} \cdot \left(-1/4\text{m} \cdot -9e^{-t/4\text{m}}\right) \\ &= 81e^{-t/4\text{m}} \end{aligned}$$

is the voltage across the inductor starting at t=0. Its initial value is therefore 81V.

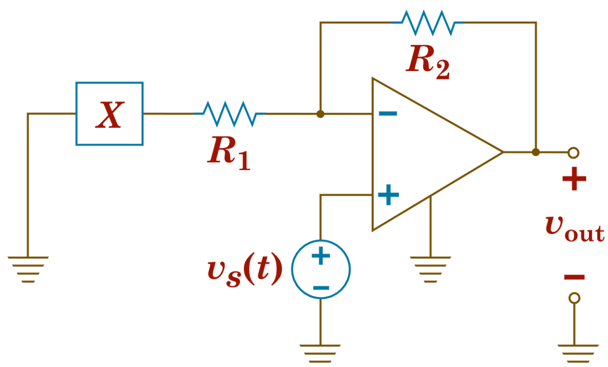
This method can be used to answer questions 1, 2, and 4 in module I2.

Question 2 might need a little explanation, though. The notation being used is that used in electronics, i.e.,



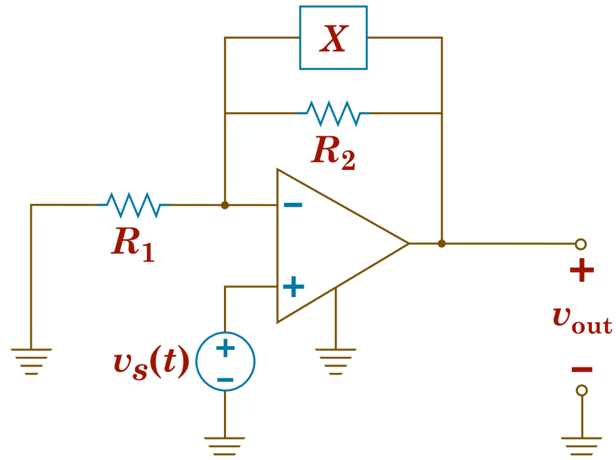
In other words, the drawing on the left is an alternative to drawing the closed loop network we are more accustomed to on the right. (There is a similar situation with a current source as well.) If you can solve the network on the right, then you can solve the one on the left.

Question 3 requires even more explanation. One version looks more or less like this:

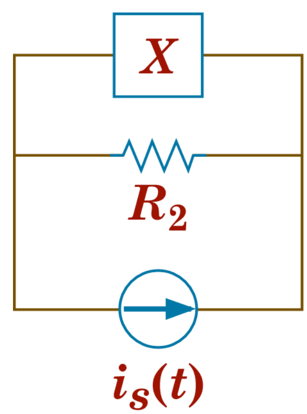


The voltages at the input terminals (+ and –) are equal to each other, so the branch containing X and R_1 has the same response as the network above with $R = R_1$. Our goal is to find the output voltage. Since we know the input voltage and the resistance R_2, all we need is the current through R_2. Since no current can flow into or out of the input terminals, all we need is to find the current in the branch containing X, which is simply another example of the same problem we have solved before.

The other version looks like this:



Unlike the previous version, we know the current through the branch containing R_1, i.e., it's a constant value. Again, since no current can flow into or out of the input terminals, that current must go to the network above the op-amp (X in parallel with R_2). Since the current is constant, it is behaving like a current source in parallel with X and R_2, as shown:



Thus, if you can find the voltage across X in this network, you can find the voltage across X in the op-amp network, which allows you to solve for the output voltage. NOTE: You might prefer to change this to its Thevenin equivalent.

Well… That's I2. As always, practice helps. If you have any questions, just let me know…