Solving Physics Problems with Multiple Representations

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Abstract: We present a teaching strategy to encourage flexible, non algorithmic problem solving. Students create several problem representations to answer questions about a single problem situation. Through reflection students learn the value of non algebraic representations for analyzing and solving physics problems.

Key words: Problem Solving, Problem Representation, Teaching Strategies, Secondary School Science, College Physics, High School Physics Curriculum

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SOLVING PHYSICS PROBLEMS WITH MULTIPLE REPRESENTATIONS

Problem solving in introductory math and science courses is often algorithmic or procedural. Students are assigned lots of problems, which they solve by mimicking the way similar problems are solved by the teacher and the textbook. Most of the assigned problems are strictly quantitative and have solutions that can be obtained by selecting and manipulating one or two appropriate equations.

Unfortunately, this type of problem solving does not result in a deep understanding of concepts nor in robust problem solving skills. There are several reasons. Students usually select an equation for a problem without checking its appropriateness and then apply the equation without seeking to understand its content. Rarely will a student reason conceptually about a problem or seek a novel solution to a problem. Most make no attempt to judge the reasonableness of their answers. True, some students do become proficient problem solvers of text-book-like problems, but the bottom line is that the majority of students come away from their introductory physics courses with a shallow understanding of concepts, and with a narrow set of problem solving skills.

At the University of Massachusetts, we have developed an approach to problem solving in introductory physics courses that puts a greater emphasis on concepts and qualitative reasoning, and that encourages students to be more flexible in how they attack problems. The approach employs a variety of strategies, which we have described elsewhere (http://www-perg.phast.umass.edu/papers/). In this note we highlight one of these strategies: *The use of multiple representations to analyze and solve problems*.

Representing a Problem in Different Ways

When reading and analyzing a problem, one forms a *representation* of the problem by interpreting it and by associating with it different pieces of knowledge. The knowledge used to represent a problem can be quite varied and may include knowledge of physical concepts and principles, equations, procedures, associated images, and related problems. How a problem is represented ultimately determines how easily the problem is solved and what is learned in the process.

Formation of a problem representation proceeds naturally as one attempts to understand and solve a problem. Nevertheless, the conscious pursuit of alternate ways to represent and solve problems does not occur naturally or easily for most people. Students need instruction and support if they are going to actively and independently try out different representations when working on problems. They must be provided with a rich set of examples of different ways to represent problem situations and then they must be asked and encouraged to use these representations to solve problems.

A simple example will help illustrate the role of representations in solving problems. Consider the problem shown in figure 1. A pumping station that will service two towns is to be built along a river bank. Where should the pumping station be positioned so that the length of pipe required to connect the pumping station to the two towns is minimal? Figure 2 contains four possible ways to represent and solve this problem. We will discuss only the first two (figures 2a and 2b).

Two towns lie along a river as shown. A pumping station is to be built along the river to supply both towns with water. Where should the pumping station be built so that the amount of pipe needed to directly connect the pumping station to the two towns is minimal?

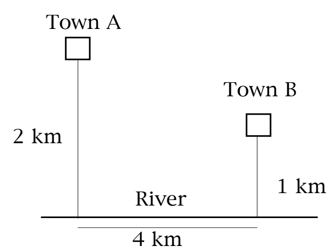

Fig. 1: Pumping Station Problem

The first representation (figure 2a) makes use of calculus and provides a textbook example of how to solve the problem formally. First a diagram is drawn portraying the situation. Elementary geometry is then used to find the total length of pipe as a function of the pumping station's position along the river. To find the position that requires the least amount of pipe, the derivative of the total length of pipe is set equal to zero. After a bit of algebra the desired position is found. (One must, of course, verify that the position found does indeed correspond to the minimum pipe length.)

In the second representation (figure 2b), Town B is transposed to the other side of the river bank. This modified problem has the same solution as the original problem because the distance from the pumping station to Town B is left unchanged by a reflection about the river bank. This new representation of the problem makes finding the solution easier. Since the shortest distance between two points is a straight line, the pumping station must be positioned along the line connecting Town A to the reflection of Town B. With some elementary geometry and a little bit of algebra the problem is solved.

Using different representations enriches the problem solving experience. Each way of representing the problem has its advantages and disadvantages. One advantage of the first representation is that it makes use of general techniques from calculus. Two disadvantages are that the representation uses fairly sophisticated mathematics, which students may not understand even if they can carry out the procedures, and the solution requires a significant amount of algebra. Further, the approach does not provide a ready interpretation of the answer. To find an interpretation requires reexamining the problem situation in light of the answer obtained.

One advantage of the second representation is that the solution is easier to recognize. In addition, the overall level of mathematics is lower and the amount of algebra is significantly less. The second representation also provides an immediate and simple geometric interpretation of the answer. Although this representation makes solving the problem easier, perceiving the value of transforming the original problem situation, and finding a better representation of the problem present a formidable challenge to students.

Using Multiple Representations to Teach Physics

Can we teach our physics students to represent and solve problems in different ways? Developing novel representations for solving problems is a creative endeavor. If each new problem required its own unique representation, teaching students to represent problems in different ways, while a worthy goal, would be impractical. Fortunately, this is not the case. In introductory physics there are a number of representations (e.g., strobe diagrams, graphs, freebody diagrams, field line diagrams etc.) that can be employed profitably in a large variety of problem situations. These representations can be used to enhance both the teaching and learning of physics.

Typically, special representations are employed in the teaching of physics in one of three modes: a) as a means to elucidate a problem, as occurs when a student draws a sketch of a physical situation and provides a summary of given information; b) as the subject of a problem, as occurs when a student is explicitly asked to draw a graph, or find the value of a physical quantity by using a graph; and c) as a step in a formal procedure, as occurs when students are required to draw freebody diagrams as one of the initial steps in applying Newton's laws to solve a problem. Each of these modes supports students' efforts to solve physics problems. They partially structure the way students approach problems, helping to shift students' attention toward something more than simply the manipulation of equations. When used consistently these three modes can lead to good habits-of-mind.

The main purpose of this note is to emphasize a fourth mode for using special representations. In this fourth mode alternate representations (i.e., other than an algebraic reprsentation) are used to analyze and solve problems. This mode is distinct from the other three in that the representation is used directly to solve a problem, rather than indirectly to support the process of solving a problem. (E.g., freebody diagrams support the application of Newton's laws). Students will not use an alternate representation to solve a problem unless they understand the representation and perceive its relative merits. One way to help students learn about alternate representations and get them to appreciate their value in solving problems is to have students solve problems using several different representations. An example is provided in figure 3.

Figure 3 displays the elements of an activity† intended for use in high school physics, early on in the study of kinematics. The activity was designed knowing that students taking introductory physics prefer to use an algebraic approach to solve kinematics problems. In the activity, students are provided with a description of a problem situation and asked to construct three different representations: strobe diagrams, equations, and graphs. The three representations are depicted in figure 4. After constructing a representation students use it to answer questions about the problem situation. After working with all three representations, students are asked to compare the different representations (e.g., which is easiest to use?, and which contains the most information?). The intent is to help students realize two important points: (a) that graphs and strobe diagrams (with a little common sense) are useful tools for analyzing the motion of objects, and (b) that for solving realistic problems (even simple ones), the algebraic representation is not always the most useful one.

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† The activity is part of the Minds-On-Physics (MOP) curriculum being developed by the Physics Education Research Group at the University of Massachusetts in Amherst. The MOP curriculum is funded by the NSF, grant ESI-9255713. A description of the MOP curriculum is available on the WWW at http://www-perg.phast.umass.edu/pages/mop.html.

We have found that in classrooms where students have some experience with strobe diagrams and graphs, students are usually able to draw and use them to correctly answer the questions raised in the activity. In contrast a large number of students are unable to make substantial progress developing and using an algebraic representation. (In fact, it is our experience that many teachers find part B of the activity challenging.) Some students do obtain answers algebraically, but often, the methods and reasoning used are faulty. In written reactions to the activity, most students express that the strobe diagrams and graphs are easiest to use and that the algebraic approach is difficult if not impossible. By no means are the reactions universal. However, generally it is the case that students who think the algebraic part of the activity is the easiest part use an overly simplistic and incorrect approach. The quotes provided in figure 5 capture some of the common sentiments expressed by students.

Several features of the problem situation make it interesting from the point of view of multiple representations. The problem situation is fairly simple. Two children race from the edge of a road to a street lamp and back. With the exception of some momentary starting and stopping, each child runs at constant speed. As a result, strobe diagrams and position graphs are reasonably straightforward to create and relatively easy to use to answer questions about the situation. What would otherwise be simple algebra, however, turns into quite complicated algebra because of two little twists: (a) The children reverse direction when they reach the street lamp; and (b) One child starts running 2 seconds after the other. The algebra, while doable, is sufficiently complex that most students will not succeed with the algebraic representation. They are not necessarily expected to succeed! We anticipate that many students will become frustrated and give up looking for an algebraic solution. This should not be taken as a sign of failure on the part of students nor on the part of the activity. Students' inability to execute the algebraic approach provides an opportunity to discuss with students how to decide when to persevere in a given approach and when to desist.

There are many lessons students can take away from activities like the one presented. So, teachers must take an active role in helping students make sense of their experiences. The issues go beyond getting students to realize that there are several ways to solve a problem. For example, how does one decide which method to use when faced with a problem? (We usually do a follow-up activity where students "try out" the different approaches while attempting to solve a set of problems specifically designed to be easier in one or another of the representations.) Teachers will find they need to push against over generalizations. The conclusion that the algebraic representation is inferior to the others should be challenged. Teachers should ensure that students get something more out of not succeeding with the algebraic representation than simply "I need to improve my algebra skills" or "that part of the activity was unreasonable." Special attention should be given to students who think that the algebraic approach is easiest, when the perceived ease-of-use is due to inappropriate reasoning on their part. Finally, emphasis should be put on the complementary nature of the different approaches. Sketching a strobe diagram or a graph can be quite informative even if one ultimately solves the problem algebraically. Graphs can be used to obtain a quick estimate of a quantity, and in many cases can be used to help find and write out algebraic relationships.

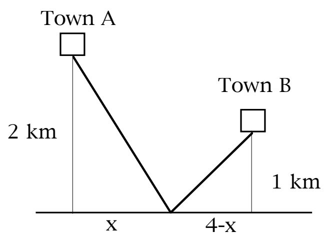
Representing a problem in different ways is a powerful problem solving heuristic. There are usually many ways to analyze and solve a problem. Analyzing a problem in different ways increases one's understanding of the problem and consequently one's chances of finding a solution. By engaging students in activities like the one in figure 3, teachers can begin to make students aware of these important lessons.

Some Final Thoughts

How can a teacher make more effective use of alternate representations? A few suggestions: **a)** Make sure that all students understand the representations you use. Explicitly teach them how to construct and interpret the common representations they are expected to know. If a student is not fluent with a representation, they are not going to consider it when faced with a difficult problem to solve. **b)** Always follow through. Have students use important representations throughout the course. Too often students are asked to learn and use a representation in one context (e.g., graphs in kinematics and freebody diagrams in conjunction with Newton's Second law), never to use it again. **c)** Elevate in importance the use of different representations <u>to analyze</u> a problem situation, and downplay the use of stock procedures <u>to get an answer</u> to a problem. Once a representation is created one can ask, and attempt to answer, a range of questions about a situation. This is well illustrated by the example in figures 3 and 4 where the graphical representation can be used effectively to answer a variety of questions about the same situation. **d)** Allow time for students to reflect upon their experiences dealing with different representations and to share their experiences with other students. Reflection provides students with an opportunity to identify and label what it is that they have learned from their own experiences. Sharing their experiences with other students gives validity to what they have learned. Together, reflection and sharing prepare the student to use what they have learned in future activities.

One goal of physics instruction should be to get students to solve problems flexibly. This requires that students consider and use multiple representations as a natural part of the way they solve problems. One must keep in mind that students will not spontaneously use alternate representations on their own. (How often do your students draw and use a graph to solve a novel problem, even after they have been asked to draw and read graphs on a number of occasions?) Engaging students in activities that explicitly require them to represent and solve problems using a variety of representations is a practical way to begin pursuing this goal.

(a) Using calculus: Let x be the distance between the pumping station and the point of the river closest to town A, and L, the total length of pipe used to connect the pumping station to the towns:



$$L = \sqrt{x^2 + 4} + \sqrt{(4 - x)^2 + 1}$$

Minimize L with respect to x:

$$\frac{dL}{dx} = \frac{x}{\sqrt{x^2 + 4}} - \frac{4 - x}{\sqrt{(4 - x)^2 + 1}} = 0$$

Solving for x:

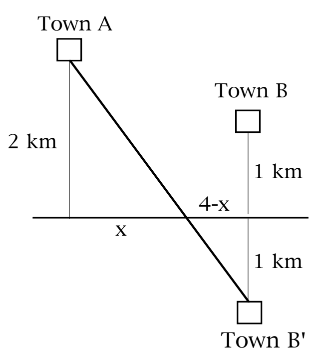
$$x^2\left((4 - x)^2 + 1\right) = (4 - x)^2\left(x^2 + 4\right)$$

$$x^2 = 4(4 - x)^2$$

$$x = 2(4 - x)$$

$$x = \frac{8}{3} km$$

(b) Using a problem transformation: Transpose town B to the other side of the river bank.

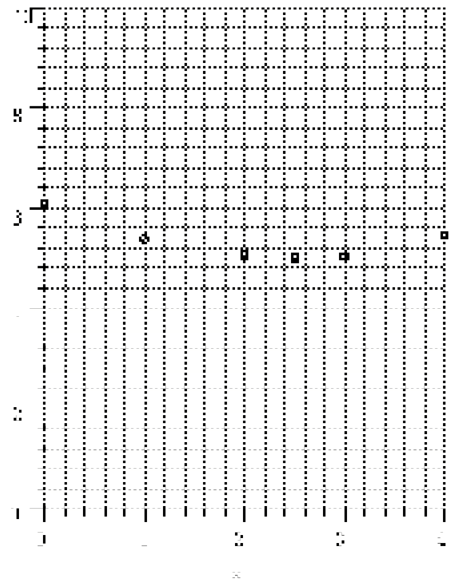


The path of shortest distance between towns A and B' is a straight line.

The triangles are similar, so: $\dfrac{x}{2} = \dfrac{(4 - x)}{1}$

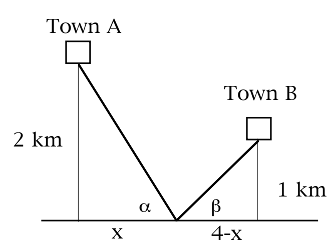
Solving for x: $x = \dfrac{8}{3} km$

(c) Using a sketch of L vs. x: A quick estimate can be gotten by plotting the total length of pipe as a function of the position of the pumping station.



An estimate of the position is $x = \dfrac{5}{2} km$

(d) Using Fermat's Principle: Pretend the edge of the river acts like a mirror. The path taken by light between any two points takes the least time (Fermat's Principle). For constant speed of light, this is also the shortest distance. For light reflecting off a mirror, the angle of incidence equals the angle of reflection ($\alpha = \beta$).

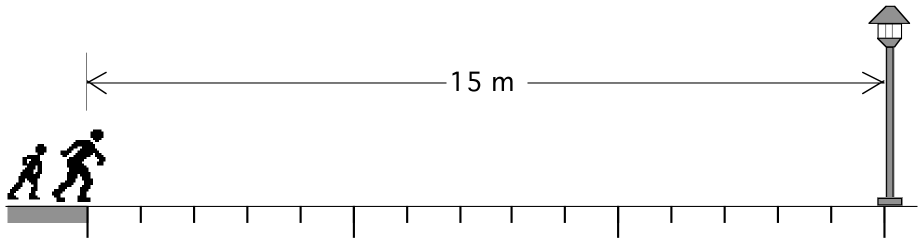


Thus the triangles are similar, so: $\dfrac{x}{2} = \dfrac{(4 - x)}{1}$

Solving for x: $x = \dfrac{8}{3} km$

Fig. 2.

Problem situation: Merinda and her little brother Joey are having a footrace from the edge of a road to a street lamp and back. At $t = 0$ seconds, Merinda starts; she runs at 2.5 m/s all the way to the street lamp and back to the starting point. Joey isn't ready at $t = 0$, and doesn't start running until $t = 2$ s; then he runs at 1.5 m/s to the street lamp and back.



Part A. Using strobe diagrams to solve problems. On the drawing above, make a strobe diagram by drawing symbols to show the positions of Merinda and Joey every second. Use your diagram to answer the following questions.

A1. About where will Joey and Merinda pass each other?

A2. Estimate the time at which this happens.

Part B. Using algebra to solve problems. Using only kinematic equations and algebra, answer the following questions about Merinda and Joey.

B1. Find the position of Merinda and Joey when they pass each other.

B2. When does this occur?

B3. Do your answers to B1 and B2 agree with your estimates from part A?

B4. Where is Joey when Merinda reaches the street lamp?

Part C. Using graphs to solve problems. Draw two position vs. time graphs, one for Merinda and one for Joey, from the time Merinda starts toward the street lamp until Joey returns to the sidewalk, then use the graphs to answer the following questions about Merinda and Joey.

C1. What is their position when they pass each other?

C2. When does this occur?

C3. Do your answers agree with your results from the previous two parts?

C4. Where is Joey when Merinda reaches the street lamp? (Does your answer agree with your result from part B?)

C5. Where is Merinda when Joey reaches the street lamp?

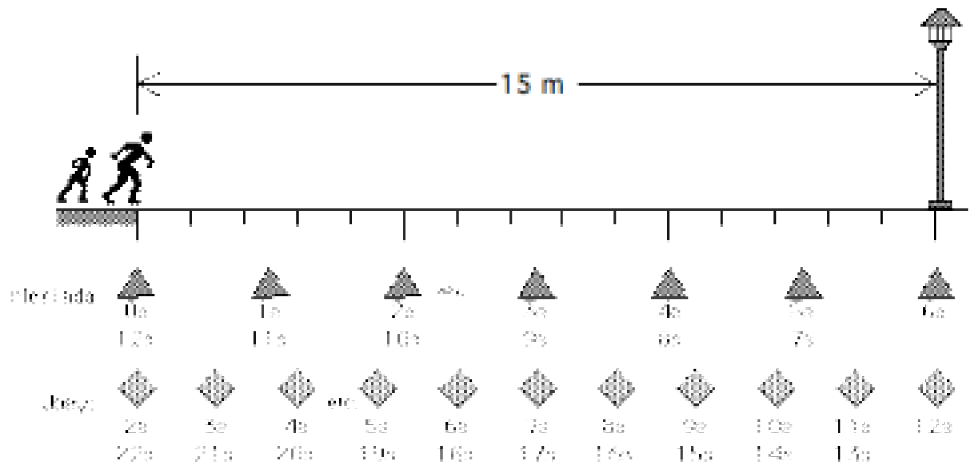
C6. How far apart are Joey and Merinda when Merinda returns to the starting point?

Reflection. When you have finished parts A, B, and C, answer the following questions.

R1. Of the three representation used in this activity, (a) which is easiest to work with? (b) which contains the most information? (c) which would you use to show someone else how to do these problems? (d) which would you like to learn better how to use? (e) which would you recommend others use to answer these types of questions?

R2. (a) Does the expression "constant velocity" as used in this activity mean that all the velocities are the same? (b) If not, how many different velocities were used, and what were they? (c) What does the expression "constant velocity" mean in this context?

Fig. 3.

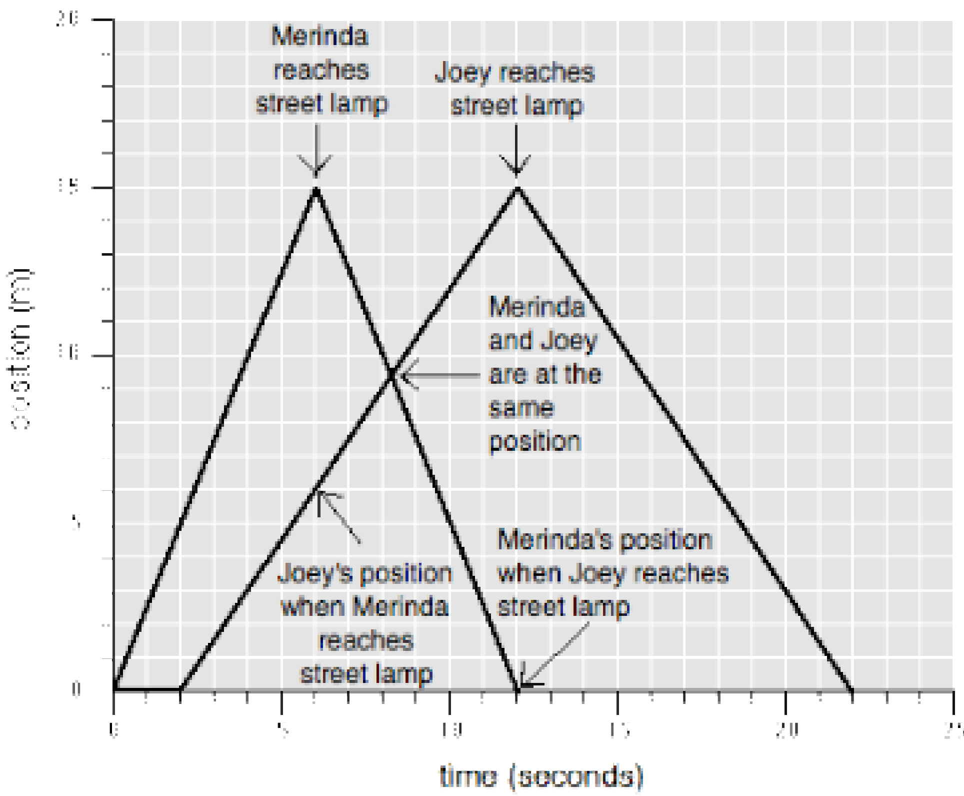


(a)

$$x_M = \begin{cases} (2.5\,\mathrm{m/s})t & 0 \le t \le 6\mathrm{s} \\ 30\,\mathrm{m} - (2.5\,\mathrm{m/s})t & 6\mathrm{s} \le t \le 12\mathrm{s} \end{cases}$$

$$x_J = \begin{cases} 0 & 0 \le t \le 2\mathrm{s} \\ (1.5\,\mathrm{m/s})(t - 2\mathrm{s}) & 2\mathrm{s} \le t \le 12\mathrm{s} \\ 30\mathrm{m} - (1.5\,\mathrm{m/s})(t - 2\mathrm{s}) & 12\mathrm{s} \le t \le 22\mathrm{s} \end{cases}$$

(b)



(c)

Fig. 4.

S1: On Activity 170 I had trouble on part B. I felt there was not enough information to do part B. Part A and Part C were very easy to understand.

S2: I liked Part C the best. I found Part A fairly easy. Part B was impossible.

S3: I liked the third part (C) better, because it gave me a picture of what was going on, and it was easier to find when their paths would cross.

S4: I thought using the graph (part C) was the easiest and the best because it gave you the clearest picture. It took me a while to figure out how to do part B, but I feel it gave me the most exact answers. Part A was not as precise because it was difficult to be exact with drawing the strobes. I liked using the graph the best.

S5: In my opinion the whole activity was difficult to understand. It's just very confusing to follow. I couldn't figure out how to do it.

S6: I found that part B was the easiest to use. I was surprised when I got the same answers for all three parts. After I graphed the points for the position vs. time I found that part C is the easiest to read the information off of. If I made this [activity] I would put part C first. I would use part B to show someone else how to do this because most people can relate to the algebra. I wish I knew how to use the algebra better.

S7: I found part A and C to be overall simple. But when I got to part B, I was lost.

S8: Part C was the easiest because it was easy to determine answers from a graph. Part B was hard because I really don't know how to use kinematic equations.

S9: The position vs. time graphs were the easiest to work with because once you drew them you [were] able to read off the positions of each person at a certain time. This was very simple. It contains the most information because the strobe and algebra method don't show 1/2 seconds and 1/2 meters as the position graphs did. I would explain how to do these problems by using the position graph. It is the most visual.

The algebra method, I thought would be the easiest, but it was the most difficult to actually picture what was happening to answer the question. I would like to learn the algebra method better.

I would recommend others use the position vs. time graphs for the clearest, most complete display of the situation.

Fig. 5.

Figure Captions

Fig. 2. Four representations of pumping station problem. (a) Calculus is used to find the position where the length of pipe is a minimum; (b) The situation is transformed into a geometrically simpler problem; (c) A graphical sketch permits an estimate of the best position and total length of pipe; and (d) Fermat's principle and the law of reflection provide an interesting connection to a seemingly unrelated physics problem.

Fig. 3. Activity from the Minds-On Physics curriculum, “Solving Constant-Velocity Problems using Different Methods”.

Fig. 4. Three different representations used to answer questions in Activity 170. (a) Strobe diagrams of the children; (b) Equations for the position of the children as functions of time; and (c) Graphs of position vs. time for the two children.

Fig. 5. Some student comments on the activity shown in figure 3. Comments were written prior to students going over the activity in class.