Theory Can Be Misleading: Balls on Tracks Revisited (Again)*

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We show an example of a theoretical calculation with hidden constraints, which make the results non-physical and misleading. We also analyze the motion of balls rolling along curved tracks. Finally, we provide some recommendations for instruction based on this exercise.

Consider a race between two balls. Both balls start on separate horizontal tracks, and have the same initial position and speed. Ball A continues along a horizontal track, while ball B descends into a valley and then rises back to its original height (see Fig. 1). Both balls cover the same horizontal distance. Which ball gets to the end of its track first? This question was analyzed in two recent *Physics Teacher* notes: Leonard & Gerace's "The Power of Simple Reasoning" [1] and Shakur & Pica's "On an Ambiguous Demonstration" [2].

Leonard & Gerace use simple motion concepts and graphs to show that ball B will always reach the end of its track first. The arguments are accessible to students, and an actual classroom demonstration of the situation is found to stimulate students' curiosity and reasoning skills. Shakur & Pica invoke a different form of argument — one based on manipulating equations derived for constant acceleration — and caution that the outcome of the classroom demonstration is ambiguous (with either ball winning the race, depending on the initial conditions). In this note, we re-assert that the apparatus of Fig. 1 is in fact a useful classroom demonstration with an unambiguous outcome: Assuming that ball B never leaves its track and that it rolls without frictional losses and without slipping at all times, ball B wins the race every time.

Simple Reasoning

Following Ref. 1, we need only focus on the horizontal component of velocity. Both balls start with the same horizontal velocity, call it v_i. For a well-constructed apparatus (tracks neither too long, too rough, nor too smooth), we can neglect energy losses. Ball A travels with a constant velocity of v_i. Focusing on the straight inclined sections of ball B's track, we see that ball B picks up horizontal velocity as it descends (since its speed is increasing) and loses horizontal velocity on the ascent, ultimately returning to exactly v_i. Ball B wins the race because its average horizontal velocity exceeds v_i (see Fig. 2).

Note that this argument works whether the ball is sliding freely (i.e., no friction) or rolling (with a static friction force exerted but doing no work). The curved sections of the track, which are discussed below, are assumed to be relatively short and therefore have been ignored in this kinematic argument.

Non-Physical Theoretical Calculation

Shakur & Pica model the track as being made up of frictionless straight-line segments, so the balls slide rather than roll, and the smooth valley of ball B becomes a V-shaped depression. The accel-

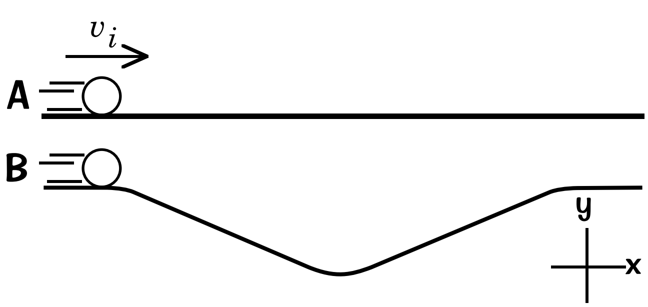

Figure 1. Set-up showing balls A and B at the start of their tracks. v_i is the speed of each ball at the beginning of the tracks.

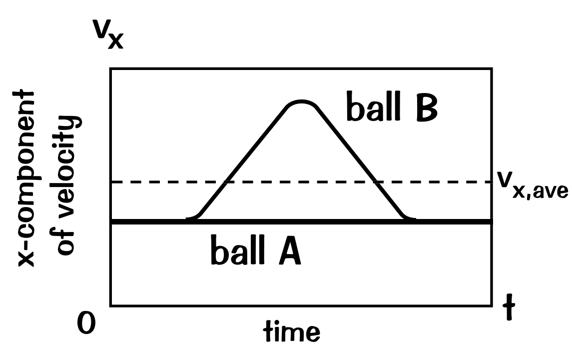

Figure 2. Plot of v_x vs. t for both balls, with average v_x indicated for ball B.

* University of Massachusetts Physics Education Research Group Internal Report PERG-1998#01-JAN#1-v.2-4pp. (1998)

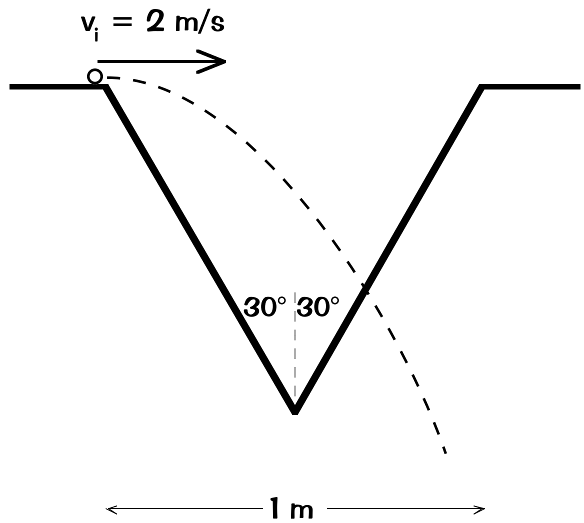

Figure 3. Smooth V-shaped track with parabolic trajectory

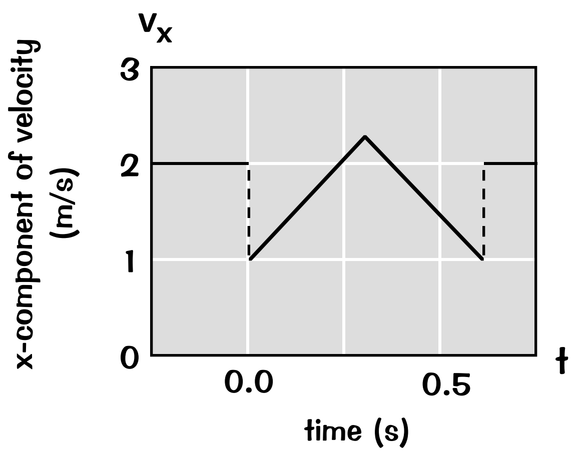

Figure 4. Plot of horizontal velocity vs. time for set-up shown in Fig. 3, assuming the ball is constrained to stay on the track

eration along each segment of track is constant and known, so simple straight-line kinematics can be used to determine how long each ball spends on each segment. Following the stipulations of the problem, both balls start with the same initial velocity v_i, and both traverse the same horizontal distance. The resulting analysis shows that ball B wins the race for small values of the initial velocity, but loses the race for large values, seeming to contradict the simple reasoning above. (See Ref. 2 for a sample calculation.) Should we distrust simple arguments? Not so quickly.

Here's the catch: The equations used by Shakur & Pica assume that ball B stays on the track at all times, even though it should leave the track after it reaches the sharp edge (kink) at the top of the incline. (See Fig. 3) This constraint requires a normal force to be directed <u>toward</u> the track at the kink point, which can dramatically reduce the horizontal velocity. (The rail of a roller coaster, for example, could provide such a constraint force.) As the initial velocity is increased, the constraint force reduces the horizontal component of ball B's velocity enough to cause ball B to lose the race. This interesting result may be relevant to a bead on a wire or a constrained roller coaster, but not to balls on tracks. If we plot the horizontal velocity vs. time (Fig. 4), we see the non-physical discontinuities caused by the constraint at the kinks. The ball instantaneously changes directions at each kink point [3].

On at least two counts the calculation does not faithfully describe the problem of balls on (even frictionless) tracks. First, a vertical jump in the velocity versus time graph implies an infinite acceleration, which is impossible. Second, the calculation imposes dynamical constraints that alter the motion from what we observe in the real world.

A Closer Look at the Curved Track Sections

To avoid discontinuities in ball B's velocity, curved sections of track are needed at the entrance, bottom, and exit of the valley. A closer look at the dynamics of a ball rolling on a curved track reveals interesting phenomena. Consider a ball of radius a that rolls down a convex circular section of track whose radius of curvature is r, as in Fig. 5.

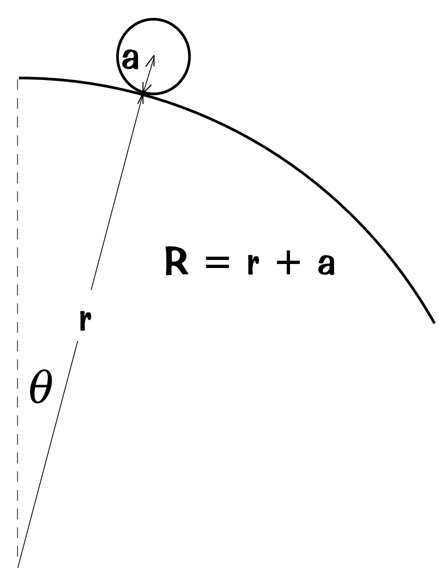

Figure 5. Ball on convex circular section of track, with r, a, θ and R labeled.

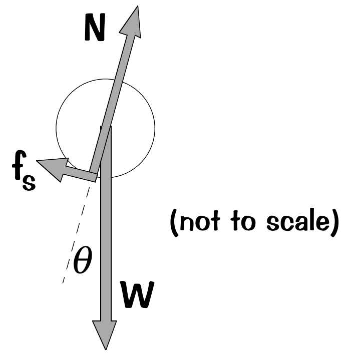

Figure 6. Force diagram for ball on curved track.

The three forces exerted on the ball are shown in Fig. 6. The gravitational force W is the only force doing work on the ball. Hence the speed of the ball's center of mass (CM) can be found at any height (or equivalently at any angle θ) by using conservation of mechanical energy. Note that the normal force N is always smaller than the normal component of the weight $W\cos\theta$, because there must be a component of the net force directed toward the center of the circular arc. Applying Newton's second law to the radial direction,

$$W\cos\theta - N = \frac{mv^2}{R}, \qquad (1)$$

where m is the ball's mass, v is the instantaneous speed of the ball's CM, and R is the radius of the CM's trajectory. As the ball moves down the track and θ increases, the normal force N must shrink in order to keep Eq. 1 satisfied.

Will the ball roll or slip?

The static frictional force f_s provides the torque needed to increase the spin of the ball as it speeds up so that it can roll without slipping. This force cannot exceed $\mu_s N$, where μ_s is the coefficient of static friction. Since N is decreasing, an angle is reached (call it θ_{slip}) where the value of f_s needed to keep the ball angularly accelerating exceeds $\mu_s N$, and the ball begins to slip. We have calculated this slip angle for the case of $\mu_s = 0.7$ (the approximate value for unlubricated steel on steel [4]), with the ball being a solid sphere (moment of inertia, $I_{\text{cm}} = {}^2/_5 ma^2$). Table I lists θ_{slip} as a function the dimensionless parameter $k \equiv v_i{}^2/gR$, where v_i is the ball's CM velocity at the top of the circle ($\theta = 0$). If $k > 1$, then gravitation is not sufficient to cause the centripetal acceleration needed at $\theta = 0$, and the ball flies off the track immediately.

Table I. Angle at which ball begins to slip as a function of the parameter $k \equiv v_i^2/gR$ for the case of a solid steel sphere ($I_{\text{cm}} = {}^2/_5 ma^2$) on a steel track ($\mu_S = 0.7$).

k (–)	θ_{slip} (degrees)
0.0	45.0
0.1	42.1
0.2	39.1
0.3	35.9
0.4	32.5
0.5	28.9
0.6	25.0
0.7	20.7
0.8	15.6
0.9	9.5
1.0	0

Note that Table I pertains to the curved section of track and not the straight inclines. It can be shown that a ball is more apt to slip on a curved track than on a straight track that is tangent to it. Hence if ball B does not slip on the curved sections of its track, it will necessarily not slip on the straight inclines.

Does the ball speed up or slow down horizontally?

As the ball rolls along the curved track in Fig. 5, does it necessarily speed up horizontally? To answer this question, one must compare the horizontal components of the normal and frictional forces. If $N\sin\theta$ exceeds $f_s\cos\theta$, then the ball will speed up horizontally. This is always the case on a straight downward incline (i.e., $r = \infty$). If the track is curved, it can be shown that, with the exception of some fine-tuned counterexamples, the ball will speed up horizontally as long as it has not yet reached its slippage limit [5]. The horizontal acceleration along the curved sections of track is manifested in the non-linear parts of ball B's v_x vs. t graph (see Fig. 2).

We emphasize that anyone wishing to build the device of Fig. 1 as a demonstration apparatus should design it so that the ball does not slip on the track. Slippage will cause some loss of kinetic energy, which could cause ball B to lose the race. More explicit design tips are included in Ref. 1.

Here are a few of the lessons we can extract from this exercise:

- **Theoretical calculations are not demonstrations.** Calculations, no matter how good they are, are based on *what we already know* about the physical world. They assume that the world behaves in a predictable way based on how well we understand physical objects. Calculations contain assumptions and approximations. Demonstrations are the real thing. Sometimes they behave in unpredictable ways, because sometimes we focus on features of the apparatus that are less relevant than others for understanding it. When this happens, the demonstration becomes a valuable learning experience, as we search for ways of understanding and explaining it.

- **Graphs can be more powerful than formulas.** We recommend a broader use of graphs in instruction and in problem solving. For example, students can use velocity vs. time graphs to visualize and interrelate an object's displacement, velocity, and acceleration, even in situations where equations would be unwieldy. Also, graphs often reveal features of a situation not manifested by equations. In this case, discontinuities in v_x vs. t reveal that we do not have a simple situation here.

- **Underlying assumptions and constraints should be made explicit.** When done correctly, calculations never give wrong answers, but they might answer the wrong questions, because the conditions and circumstances imposed or assumed might not correspond to what was intended. Students can become confused when assumptions are left unstated, and often they are not aware of the assumptions they are making when analyzing a new situation. In the case of this theoretical calculation, we have seen that unintended constraints on the motion of balls on tracks can lead to results that are erroneous, not just quantitatively, but qualitatively.

- **Contradictions should be fuel for further investigation.** Sometimes two theoretical calculations seem contradictory. At other times, theoretical predictions are contradicted by empirical observation. Such discord should prompt students to examine their models and assumptions more closely in order to resolve the difficulty.

For most of us, the beauty of physics is its simplicity, its deductive power, and its faithfulness to the real world. There is a delicate balance among these characteristics. Too simple a model will not accurately describe reality, while taking into account all the details of a physical system renders the description too complicated. Demonstrations are a wonderful opportunity for all of us — students and teachers — to discover if our models of the real world are simple enough to be useful and complicated enough to actually predict the behavior of something. We learn which features of a situation can be ignored, and which features cannot! We become more aware of hidden assumptions and of how we think. In other words, we learn how to reveal, challenge, and modify our models of the real world.

References

[1] William J. Leonard and William J. Gerace, *Phys. Teach.* **34**, 280 (1996).

[2] Asif Shakur and Andrew Pica, *Phys. Teach.* **35**, 316 (1997).

[3] Note that there is also a discontinuity in v_y vs. t when the ball reaches the kink at the bottom of the V-shaped track.

[4] D.C. Giancoli, *Physics Principles with Applications*, 4th Ed. (Prentice Hall, Englewood Cliffs, NJ, 1995), p. 93.

[5] Work in preparation by the authors.