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Random Means Hard to Digest

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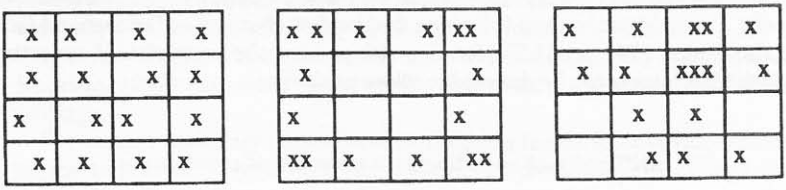
Randomness is one of the key concepts of probability theory and statistical inference (Falk & Konold, 1992). We all feel we know what we mean when we speak of "chance" and "randomness". These concepts often serve us well in scientific and everyday communication because of a general consensus about their meanings (Falk, 1991; Green, 1989; Noether, 1987). Yet, randomness is one of the most elusive concepts in mathematics. It resists easy or precise definition (Zabell, 1992). Nor can it be established for certain whether a particular sequence is "truly random" (Ayton, Hunt, & Wright, 1989; Chaitin, 1975; Steinbring, 1991). Hacking (1965) was therefore a bit optimistic when he suggested that the meaning of the word "random" could be "... answered briefly, but it would take 100 pages to prove any answer correct" (p. 118).

The perplexing nature of randomness poses an educational dilemma. Shall we forgo discussing the meaning of the concept in our teaching (relying on students' existing intuitions)? Or shall we endeavor to find a satisfactory way of presenting randomness, undertaking the challenge of bringing up the doubts and difficulties students will predictably have?

One possible solution is to introduce students to a meaning of randomness that is both intuitively acceptable and agrees with the standard concept of randomness as a process characterized by statistical independence (and random sequences as the products of that process). Such an interpretation of randomness is based on the mathematical concept of *complexity* (Chaitin, 1975; Fine, 1973, chap. 5). Despite sophisticated computational techniques, the idea itself is amazingly simple: a *random* sequence[1] is one which *cannot be condensed*, that is, it cannot be described by a program that is substantially shorter than itself. A random sequence has, in fact, to be specified almost symbol by symbol.

Another look at Green's randomness tasks

To demonstrate the complexity approach to randomness, we consider two interesting problems used by David Green. He has used these in both his research and his classes as a trigger for a teacher-guided discussion. Green (1979, 1989) showed children a drawing of a 4x4 square roof tiled by 16 unit squares. They were asked to imagine that 16 snowflakes had fluttered down onto the roof, and were given the Piagetian task of marking 16 *x*s on the drawing to show where they thought each snowflake would land. Figure 1 shows three distributions that, according to Green, children typically generate.



(a) (b) (c)

Figure 1. Typical children's drawings of snowflake distribution (Green, 1979, 1989).

The following are characteristic verbal descriptions children give these (respective) drawings:
(a) one per square,
(b) all around the edge,
(c) a random distribution.
One answer to the question of what "random" means is hidden in these descriptions. Note that statements (a) and (b) are effective descriptors of the location of the *x*s in the respective drawings. Indeed, if we ignore location within a cell, distribution (a) can be reproduced exactly from the verbal description. It would, however, be difficult to think of any shorthand description of distribution (c) that could be used to communicate the location of the various *x*s. Disregarding the question of whether the distribution in (c) will pass standard tests of randomness, we note that its description "random distribution" is a shorthand expression that is used to replace an enumeration of the contents of each cell, and it conveys the idea "there is no easy way to describe this layout".

In another problem, Green (1987) used the following binary sequences, each comprising 20 symbols. Students were told that each sequence was produced by a machine designed to mimic coin flipping. Their task was to decide which of the machines (sequences) was fair (random) and which was biased (nonrandom).

1. H T H T H T H T H T H T H T H T H T H T
2. H H H H H H H H H H T T T T T T T T T T
3. H H T H H T H H T H H T H H T H H T H H
4. H T T H H T T T T H T T H T T T T H T T

Green expects that only sequence 4 would be judged random, whereas the first three are obviously patterned (i.e., nonrandom). Our personal experience accords with Green's statement: while typing the four sequences for this article we noted that it was quite easy to copy the first three sequences. We gave each of them one quick look and successfully copied it in one attempt. In contrast, the reproduction of sequence 4 required significantly more time and careful attention, with frequent viewings of the original sequence. Indeed, it was typed in several "chunks" of about five characters each.

The characterization of a random sequence as one that is difficult to compress or to reproduce captures well the conception of randomness as *unpredictability*. Unlike sequence 1, which can be condensely described by HTHT..., the meaning of "..." would *not* be clear following the first few characters of sequence 4 (see also Paulos, 1991, pp. 47-51). Random strings cannot be extrapolated because no valid lawfulness can be detected to allow prediction.

Difficulty of encoding as a measure of randomness

The view of randomness as incompressibility has its heritage in information theory, a science developed mainly since World War II which studies the transmission of messages. Its definition of randomness is based on the observation that information embodied in a random series of symbols cannot be compressed or reduced to a more compact form. More precisely, a series of characters is random if the shortest computer program capable of reproducing it has about the same number of bits as the series itself. The length of the shortest program is called the sequence's *complexity*. It measures, in fact, *difficulty of encoding* and serves to quantify the degree of randomness of (and information contained in) the sequence.[2]

When the complexity measure is used to sort long sequences into random and nonrandom (based on some reasonable predetermined cut off point), the results are, in general, compatible with those of statistical tests for randomness (which are also based on a predetermined "level of significance"). A detailed exposition of the relation between the complexity and the statistical definitions of randomness can be found in Fine (1973).

Perceived randomness and difficulty of encoding

The complexity interpretation of randomness has lately found its way into philosophical and psychological writings (e.g., Attneave, 1959; Dennett, 1991; Falk, 1975; Garner, 1970; Simon, 1972) as well as into the popular mathematical literature (Paulos, 1991). It has, however, received limited attention as a possible *psychological variable* in accounting for *subjective judgment of randomness*. Kahneman and Tversky (1972) have suggested that random-appearing sequences are those whose verbal description is longest, and that apparent-randomness is a form of complexity of structure. Still, to the best of our knowledge, no research relating difficulty of encoding to perceived randomness has been pursued by statistics-educators or psychologists.

The most prominent and consistent finding of the psychological research on the perception of randomness of binary sequences (and two-dimensional tables) is that people identify randomness with an excess of alternations between the (two) symbol types (Falk, 1975, 1981; Lopes & Oden, 1987;

Wagenaar, 1972). Sequences that are prototypically random (e.g., they include the modal number of runs) are not perceived by subjects as maximally random because the runs seem subjectively too long to be random. The well-known gambler's fallacy (i.e., the belief that tails is more probable than heads after a sequence of successive heads) is, in fact, equivalent to the belief that symbol types in a random sequence should show frequent alternations. Similar results are reported when subjects are asked to generate or to simulate random sequences (Bakan, 1960; Falk, 1975, 1981; Noether, 1987; Tune, 1964; Wagenaar, 1972), although recent studies (Kareev, 1992; Rapoport & Budescu, 1992) cast some doubt on the consistency and meaning of these findings.

For every finite binary sequence, one can determine the relative frequencies of the two symbol types and the conditional probability of change (or continuity) after a given symbol. If there are n symbols in the sequence, the number of opportunities for a change of symbol type is n-1. For example, the sequence below comprises 21 characters. The number of possible changes in symbol type is 20, while the number of actual changes in this sequence is 12, as depicted.

```
X O X X O X O X O O X O O O O X O X X X X
 : :   : : : : :   : :       : : :
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The overall *probability of alternation* of the sequence, denoted P(A), is obtained by dividing the number of actual changes in symbol type in the sequence by n-1. In our example $P(\text{A})=12/20=0.6$.

When the frequencies of the two symbol types in a long, random sequence are *equal*, P(A) should be close to 0.5. Finite sequences with P(A) values other than 0.5 do occur, but they are less likely. Sequences deviating from $P(\text{A})=0.5$ are equally probable in the two directions. This means, for example, that a sequence with $P(\text{A})=0.7$, which contains more alternations than expected, has about the same probability of occurring as a sequence with $P(\text{A})=0.3$, in which there are fewer alternations (longer runs) than expected. It has been demonstrated, however, that people generally regard sequences with $P(\text{A})=0.7$ as more likely to occur by chance than sequences with $P(\text{A})=0.3$.

The function of apparent-randomness was obtained by Falk (1975, 1981). Subjects were shown a set of 10 sequences of the same length (21). The sequences comprised two symbol types whose frequencies in the sequence differed by one. Their probabilities of alternation ranged from 0.1 through 0.2, 0.3, to 1.0 (a perfectly alternating sequence). Subjects were asked to rate each sequence for randomness.[3] *Apparent-randomness (AR)* was measured by the mean randomness rating of 219 subjects. *AR*, as a function of P(A), peaked at 0.6, instead of at 0.5, and it was negatively skewed (see Figure 2). As mentioned, sequences with P(A)s equally distant from 0.5 were *not* judged equally random.

The apparent-randomness function is compared in Figure 2 with an "objective randomness" function of the sequences as measured by their second-order *entropy*[4] *(EN)*. The *EN* function is symmetric about $P(\text{A})=0.5$ where it peaks. There is a marked discrepancy between the two functions. Subjects' ratings of randomness do *not* closely match the objective randomness of the sequences. The correlation coefficient between the sequences' entropy measures and their mean rated randomness is only 0.54.

While it has been convincingly established that subjects show consistent biases in judgments of randomness, the explanations for these biases are, in our view, not as convincing. Various accounts have been suggested by psychologists for subjects' performance in tasks of judgment and generation of randomness. These include descriptions of response tendencies (Tune, 1964) and of several variables affecting perception (Wiegersma, 1987). Kahneman and Tversky (1972) have explained these results in terms of the heuristic of *local representativeness*.

According to Kahneman and Tversky's account, subjects regard a sequence as random if it reflects the salient features of its parent population. Thus, when generating a sequence to simulate successive coin-flipping outcomes, subjects balance the frequencies of heads and tails to conform to the long-run expected proportions. They try, however, to equate the relative frequencies of the two outcomes not only *globally*, in the long sequence, but also *locally*, in short subsequences. They regard chance as a self-correcting mechanism, which promptly takes care to restore the balance whenever it is disrupted. This results in exaggerated alternations and shorter runs than typically found in chance productions. They describe subjects as applying the law of large numbers too hastily, as if they believe in "the law of small numbers" (Tversky & Kahneman, 1971).

Gigerenzer (1991) has recently criticized Tversky and Kahneman's program of "heuristics and biases" in judgment under uncertainty on several grounds. One of Gigerenzer's claims is that often the heuristically-based accounts of biases offer little more than a redescription of the phenomena they purport to explain. Although local representativeness is an insightful and tenable redescription of subjects' performance in randomness tasks, its predictive ability is rather limited. Indeed, one can infer from local representativeness that overalternations are expected, but the extent of that bias cannot be predicted.[5] The heuristic specifies neither *how local* subjects' span of consideration is nor *how representative* that local subsequence is supposed to be.

We suggest an alternative explanation, that subjects base their judgment on a subjective assessment of the complexity of the sequence. When asked to rate the randomness of a sequence, they evaluate the difficulty of an attempt to memorize, reproduce, or concisely encode it. The harder that (implicit) task, the more random the sequence is judged to be. Our hypothesis would be supported if perceived randomness was better predicted by the sequence's difficulty of encoding (i.e., subjective complexity) than by its entropy (i.e., objective randomness).

The variable "difficulty of encoding" which we suggest accounts for perceived randomness, is a subjective variable. It may, however, supply an indirect answer to the question how local and how representative a sequence should be in order to appear maximally random. The answer is: *so as to make it hardest to encode*. The hypothesis that subjective complexity may predict apparent-randomness, if supported, would be valuable in partly explaining subjects' mechanism of assessing randomness.

One measure of subjective complexity might be the time it would take to accurately copy a sequence. If our hypothesis is true (a) copying a sequence of P(A)=0.6 would require more time than copying a sequence of P(A)=0.5, and (b) the apparent-randomness function would be more closely correlated with the durations of copying the sequences than with normative measures of randomness (such as entropy) of these sequences.

We are now conducting research in which subjects are required either to transcribe, memorize, or dictate the sequences, and are developing various experimental indices of *difficulty of encoding* to correlate with the sequences' judged randomness. In a preliminary study, 10 subjects were presented with the same sequences for which apparent-randomness has been plotted in Figure 2. They were instructed to inspect each sequence until they could write it from memory. If they erred, they were given more time to memorize the sequence until they managed to reproduce it in one attempt. Memorization time *(MT)* up to the first successful performance was recorded for each sequence. The 10 *MT* measurements of each subject were standardized. For each P(A), we computed the mean of the standard scores over the 10 subjects. This mean is our measure of *difficulty of encoding (DE)* of the sequence.

Figure 2 presents *DE* as a function of P(A) alongside the apparent-randomness and the entropy functions.

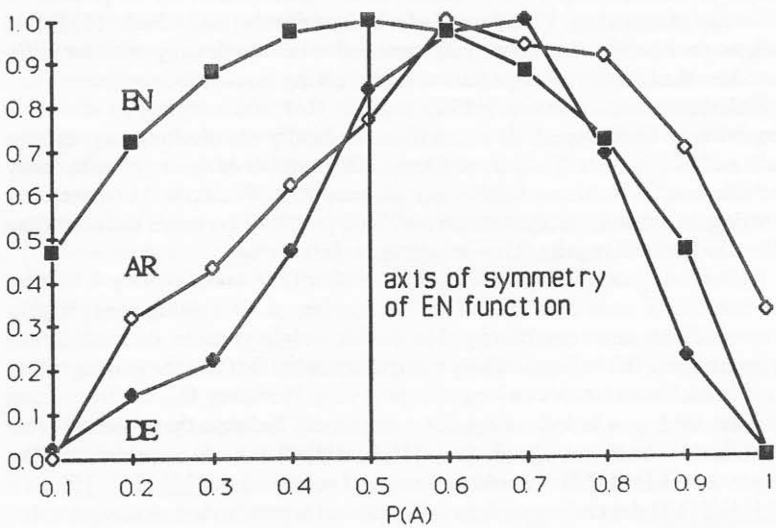

Figure 2. Apparent-randomness, difficulty of encoding, and entropy as functions of the sequence's probability of alternation, *P*(A). (*AR* is apparent-randomness, linearly transformed; *DE* is difficulty-of-encoding, linearly transformed; *EN*, second-order entropy, is a measure of objective randomness.)

The entropy values range from 0 to 1. Apparent-randomness was rated on a scale from 1 (not at all random) to 20 (perfectly random). Subjects' mean ratings ranged from 3.0 to 14.6. *DE*, as a mean of standard scores, ranged from -1.049 to 1.244. *AR* and *DE* were linearly transformed to range from 0 to 1 to permit comparisons among the three functions.

Figure 2 indicates that, as hypothesized, the apparent-randomness function is closer to the function of difficulty of encoding (subjective complexity) than to that of entropy (objective randomness). The correlation between *AR* and *DE* is 0.89, whereas the correlation between *AR* and *EN* is 0.54. Difficulty of encoding turned out maximal for *P*(A)=0.7 rather than for 0.5, and it was greater for 0.6 than for 0.5. This supports the hypothesis that subjective complexity mediates the judgment of randomness. It should be recalled, however, that *DE* was not obtained by rating a sequence's perceived complexity. *DE* is a performance variable meant to quantify subjective complexity.

Predicting subjective complexity

What is it that subjects are doing when they memorize sequences that makes some sequences more difficult than others? To address this question, we attempted to find a simple, but objective, way of quantitatively describing the subjective complexity of a sequence that would permit us to predict its difficulty of encoding. We adopted a technique developed by Falk (1975) that assigns each sequence a numerical score, called a *complexity predictor (CP)*, based on the number of pure runs and alternating runs in the sequence.

Kahneman and Tversky (1972) suggest that when trying to dictate a sequence of binary symbols one will undoubtedly use shortcut expressions such as "four Ts," or "H-T, three times." The number of these "chunks" may provide one indication of difficulty of encoding. We note, however, that forming the chunk "H-T, three times" will probably be more difficult than "four Ts." It will require more counting and checking.

Hence, our procedure quantifies the complexity of a sequence by summing the number of pure runs and twice the number of alternating runs. Higher scores indicate more complexity. The double weight given to alternating runs in determining the value of *CP* depends partly on the fact that the unit repeating itself in such runs is twice as long as in pure runs. However, this criterion alone will not lead to a unique score for a sequence, because there are no clear boundaries between pure and alternating runs. For instance, the sequence H H H T H T could be assigned scores of 4 (H H H T H T) or 3 (H H H T H T) depending on how the sequence is partitioned. A unique value does exist, however, if we agree to partition each sequence to achieve the lowest possible score (in this case, 3). Thus, for example, the sequence below, with *P*(A)=0.2,

H H H H H H T T T H H T T T T T T T T H H H

was scored by listing 5 uniform runs, as depicted. The sequence's *CP* would therefore be 5. In contrast, the following sequence, with *P*(A)=0.8,

H H T H T H T H T H T T T H H T H T H T H

was scored by listing 5 chunks, two of which are doubly weighted because they are runs of alternations. The *CP* measure of this sequence would thus be 7. This may explain why, although these two sequences deviate equally from the expected $P(\text{A})=0.5$, the one with overalternations is empirically rated as more random.

To test the adequacy of this method, we calculated the "complexity predictor" *(CP)* for the sequences used by Falk (1975). The procedure gave higher *CP* scores to sequences with exaggerated alternations ($P(\text{A})>0.5$) than to sequences with the same extent of exaggeration in uniform runs ($P(\text{A})<0.5$). Additionally, some sequences with exaggerated alternations received higher values for *CP* than sequences with $P(\text{A})=0.5$. The correlation coefficient between *CP* and *AR* across the sequences was 0.90, as compared with 0.54, the correlation between entropy and *AR*.

These results suggest that sequences with overalternations are perceived as more random than their entropy warrants because of a relative difficulty in processing successive alternations compared with uniform runs. *CP* is a rough index of subjective complexity; it addresses only sequential dependencies conditioned on one preceding symbol (as does the *EN* measure), it depends on somewhat arbitrary weights, and is based on one particular partition of the sequence, not necessarily the same partition used by subjects. Yet it is highly correlated with apparent-randomness.

Furthermore, *CP* matches the subjects' difficulty-of-encoding function nicely. The correlation coefficient between *CP* and *DE* was 0.96. This suggests that whatever strategies subjects employ in memorizing the sequences, the difficulty of the task is strongly related to the above weighted sum of runs. The same is apparently true for the judgment of randomness.

From a mathematical point of view, a run of alternations is as redundant as a uniform run. Both allow perfect prediction within the boundaries of the run. The conditional probability of change following a given symbol is 1 in the former and 0 in the latter. Psychologically, however, they appear not to be equivalent. This may be the root of the bias in regarding sequences with too many alternations as maximally random, and of the negative skewness of the *DE* and *AR* functions.

Implications for teaching

The teaching of probability and statistics relies heavily on the concepts of chance and randomness. Statistical educators should therefore be aware of the "theories" and preconceptions concerning these concepts that students possess before receiving any instruction. The existing psychological research on peoples' biases and misperceptions of randomness is thus highly relevant for educators. The similarity between the concepts of randomness and complexity, and our preliminary findings of a high correlation between subjective randomness and subjective complexity, may shed more light on the psychology of randomness.

Whether people's perception of randomness of sequences is determined by their expecting locally-representative examples of the generating rule, or they infer randomness from their experience of complexity, they end up identifying randomness with excessive alternations. Perhaps both processes work jointly and converge on the same result. Students' misperceptions of randomness might thus be psychologically overdetermined. Teachers should therefore realize that there are powerful reasons behind these robust misperceptions. They cannot be easily eradicated. As a prerequisite for change, however, students and teachers ought to know something about how we are already thinking of randomness. Being aware of our biases, and understanding the mechanisms accounting for them, may be a first step toward overcoming them.

Since it appears that in judging randomness, subjects attend to the complexity of sequences[6], it might be possible to foster a more intuitive, yet mathematically sound, conception of randomness if it is introduced via the complexity interpretation. Certainly, we do not mean to suggest that beginners be instructed in the technicalities of information theory and coding schemes; rather, we suggest that the relation between the theoretical concepts of randomness and complexity be made more explicit. To make the idea concrete, it may be of instructional value to have students copy several sequences with varying P(A)s. They may note that the harder it is to copy a sequence, the more random it appears. A definition of randomness based on objective difficulty of encoding (i.e., complexity) will thus have intuitive appeal. Students will certainly understand it, in particular if they try to transcribe, dictate, or memorize several such sequences. This introduction may help students come to grips with the meaning of randomness.

Clearly, introducing randomness through the sequences' difficulty of encoding will *not* help in remedying the gambler's fallacy. On the contrary, (subjective) difficulty of encoding, as a function of P(A), displays the same shift relative to objective randomness as the function of perceived randomness. This, in fact, might be the reason for people's misperceptions of randomness.

The main advantage of introducing the complexity interpretation lies in the insight students may gain concerning the meaning of randomness of *sequences*. The mathematical definition of randomness as statistical independence is meaningful with respect to the *process* that generates the sequences. The outputs of that process, however, defy a simple determination of their randomness. Attempting to encode such sequences and appraising the difficulty of this assignment could fill this gap. It may provide an intuitive assessment of the extent of randomness of sequences as a first approximation. This intuitive assessment should, however, be somewhat adjusted, because difficulty of encoding (just like apparent-randomness) is subject to small systematic biases.

REFERENCES

Attneave, F. (1959). *Applications of information theory to psychology: A summary of basic concepts, methods, and results.* New York: Holt, Rinehart & Winston.

Ayton, P., Hunt, A.J., & Wright, G. (1989). Psychological conceptions of randomness. *Journal of Behavioral Decision Making, 2*, 221-238.

Bakan, P. (1960). Response tendencies in attempts to generate random binary series. *American Journal of Psychology, 73*, 127-131.

Chaitin, G. J. (1975). Randomness and mathematical proof. *Scientific American, 232*, 47-52.

Dennett, D. C. (1991). Real patterns. *The Journal of Philosophy, 88*, 27-51.

Falk, R. (1975). *Perception of randomness.* Unpublished doctoral dissertation, Hebrew University, Jerusalem (Hebrew with English abstract).

Falk, R. (1981). The perception of randomness. In *Proceedings of the Fifth International Conference for the Psychology of Mathematics Education* (pp. 222-229). Grenoble, France.

Falk, R. (1991). Randomness -- an ill-defined but much needed concept (commentary on "Psychological Conceptions of Randomness"). *Journal of Behavioral Decision Making, 4*, 215-218.

Falk, R., & Konold, C. (1992). The psychology of learning probability. In F. S. Gordon & S. P. Gordon (Eds.), *Statistics for the twenty-first century* (pp. 151-164). The Mathematical Association of America.

Fine, T. L. (1973). *Theories of probability: An examination of foundations.* New York: Academic Press.

Garner, W. R. (1970). Good patterns have few alternatives. *American Scientist, 58*, 34-42.

Gigerenzer, G. (1991). How to make cognitive illusions disappear: Beyond "heuristics and biases." *European Review of Social Psychology, 2*, 83-115.

Green, D. R. (1979). The chance and probability concept project. *Teaching Statistics, 1*, 66-71.

Green, D. (1987). Probability concepts: Putting research into practice. *Teaching Statistics, 9*, 8-14.

Green, D. (1989). School pupils' understanding of randomness. In R. Morris (Ed.), *Studies of mathematics education, Vol. 7: The teaching of statistics* (pp. 27-39). Paris: UNESCO.

Hacking, I. (1965). *Logic of statistical inference.* Cambridge: Cambridge University Press.

Kahneman, D., & Tversky, A. (1972). Subjective probability: A judgment of representativeness. *Cognitive Psychology, 3*, 430-454.

Kareev, Y. (1992). Not that bad after all: Generation of random sequences. *Journal of Experimental Psychology: Human Perception and Performance, 18*(4), 1189-1194.

Lopes, L. L., & Oden, G. C. (1987). Distinguishing between random and nonrandom events. *Journal of Experimental Psychology: Learning, Memory, and Cognition, 13*, 392-400.

Noether, G. E. (1987). Mental random numbers: Perceived and real randomness. *Teaching Statistics, 9*, 68-70.

Paulos, J. A. (1991). *Beyond numeracy: Ruminations of a numbers man.* New York: Alfred A. Knopf.

Rapoport, A., & Budescu, D. V. (1992). Generation of random series in two-person strictly competitive games. *Journal of Experimental Psychology: General, 121*, 352-363.

Simon, H. A. (1972). Complexity and the representation of patterned sequences of symbols. *Psychological Review, 79*, 369-382.

Steinbring, H. (1991). The concept of chance in everyday teaching: Aspects of a social epistemology of mathematical knowledge. *Educational Studies* *in Mathematics, 22*, 503-522.

Tune, G. S. (1964). Response preferences: A review of some relevant literature. *Psychological Bulletin, 61*, 286-302.

Tversky, A., & Kahneman, D. (1971). Belief in the law of small numbers. *Psychological Bulletin, 76*, 105-110.

Wagenaar, W. A. (1972). Generation of random sequences by human subjects: A critical survey of literature. *Psychological Bulletin, 77*, 65-72.

Wiegersma, S. (1987). The effects of visual conspicuousness and the concept of randomness on the recognition of randomness in sequences. *The Journal of General Psychology, 114*, 157-165.

Zabell, S. L. (1992). The quest for randomness and its statistical applications. In F. S. Gordon & S.P. Gordon (Eds.), *Statistics for the twenty-first century* (pp. 139-150). The Mathematical Association of America.

AUTHORS' NOTES

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FOOTNOTES

[1] To simplify matters, we refer to a binary sequence, but the arguments apply as well to a matrix or to any other spatial layout of two or more symbol types.

[2] Defining complexity in terms of the length of the shortest algorithm for a digital computer raises a problem: which computer shall be employed? Likewise, what particular computer language should be used? Different machines, communicating through different computer languages, might require more or fewer bits when instructions are translated from one to the other. Actually, however, the choice of computer and language matters very little. The problem may be avoided by insisting that the randomness of all sequences be tested on the same machine (Chaitin, 1975).

[3] In fact, about half the subjects rated how likely such a sequence was if the cards presenting the two symbol types had been perfectly shuffled beforehand. The other half rated how likely it was, given the sequence, that the cards had been well shuffled. There was no difference between the two groups' responses.

[4] A sequence's entropy index is an estimate obtained by the use of empirical proportions which replace the "true" population probabilities. The definition and method of computation of the entropy measure can be found in Attneave (1959, pp. 19-21).

[5] Kareev (1992) has suggested an interesting hypothesis that can predict results of generation of randomness (but not of perception of randomness). His predictions are based on the local representativeness heuristic curtailed by the limitations of short-term memory.

[6] This statement is based on powerful introspective data, and is inferred from our pilot results despite risking application of "the law of small numbers". (Our continued investigations further support that contention.)