NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

Representing Probabilities with Pipe Diagrams
Author(s): Clifford Konold
Reviewed work(s):
Source: *The Mathematics Teacher*, Vol. 89, No. 5 (MAY 1996), pp. 378-382
Published by: National Council of Teachers of Mathematics
Stable URL: http://www.jstor.org/stable/27969794
Accessed: 04/03/2013 13:40

Your use of the JSTOR archive indicates your acceptance of the Terms & Conditions of Use, available at
http://www.jstor.org/page/info/about/policies/terms.jsp

JSTOR is a not-for-profit service that helps scholars, researchers, and students discover, use, and build upon a wide range of content in a trusted digital archive. We use information technology and tools to increase productivity and facilitate new forms of scholarship. For more information about JSTOR, please contact support@jstor.org.

National Council of Teachers of Mathematics is collaborating with JSTOR to digitize, preserve and extend access to *The Mathematics Teacher*.

http://www.jstor.org

This content downloaded on Mon, 4 Mar 2013 13:40:01 PM
All use subject to JSTOR Terms and Conditions

Clifford Konold

Representing Probabilities with Pipe Diagrams

Probability is a notoriously difficult concept. Even after much instruction, many students remain confused both about methods used to calculate a probability and about its meaning (Konold 1991). In this article, I shall describe a modified version of the tree diagram that many of my students at both the high school and college levels have found helpful in making probabilities more meaningful. I refer to these representations as *pipe diagrams.* Although most readers are probably familiar with tree diagrams, I review a few of their basic features before introducing pipe diagrams and discussing their educational advantages.

A college textbook on finite mathematics by Kemeny, Snell, and Thompson (1956) includes the earliest example of using tree diagrams to compute probabilities that I could find. The authors first describe the tree diagram as a "useful tool for analyzing logical possibilities" (p. 25) and then explicitly use tree diagrams to define a measure of, and compute probabilities associated with, series of independent events (pp. 140–42).

Many students are confused about how to calculate a probability

Figure 1 is a tree diagram representing the four possible outcomes of flipping a coin twice. On the first flip, the coin can land either heads, H, or tails, T. These possibilities are represented by the two branches on the left in **figure 1,** one leading to H, the other to T. The probability of each occurrence, 1/2 in this example, is traditionally written next to the branches, as shown. Regardless of what happens on the first flip, it is possible to get either H or T on the second flip. This outcome is represented by splitting each original branch into two branches, one labeled H and the other, T. Because the outcome of the second flip is independent of the outcome of the first flip, the branches in the second level are still tagged with probability 1/2. Thus, the 1/2 on the top branch is the value of P(H/H), the *conditional* probability of getting H given an H on the first flip. The completed tree diagram has four unique paths through the branching system. Each path corresponds to one of the four possible events: HH, HT, TH, and TT.

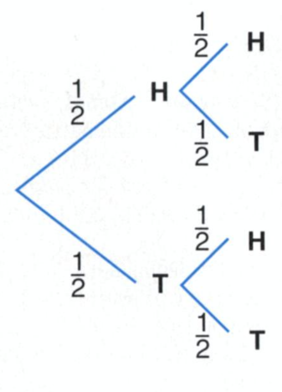


Fig. 1
Tree diagram for two flips of a fair coin

Probabilities of a variety of events can be computed using the tree diagram. For example, the simplest way to compute P(HH), the *joint* probability of getting H on both flips, is to notice that because H and T are equally likely for each flip, the four paths in the tree diagram must each have the same probability, 1/4. However, students familiar only with this method may not be able to compute joint probabilities when they cannot readily construct the tree diagram—for example, the probability of twenty heads in a row—or when the elementary outcomes are not equally probable—for example, the probability of two Hs with a coin "loaded" such that P(H) equals 0.6. In the situation of the biased coin, the four possible events are no longer equally likely, and thus determining the probability of two Hs as a fraction of the total number of paths would be inappropriate.

The more general method for computing a joint probability from a tree diagram is to multiply the probabilities of the individual outcomes. Thus, P(HH) is the probability of taking the first H-branch, 1/2, multiplied by the conditional probability of taking the second H-branch, or (1/2) times (1/2).

Clifford Konold teaches at the University of Massachusetts, Amherst, MA 01003, and is also codirector of the Math Center at TERC, Cambridge, MA 02140. He has been developing probability and data-analysis curricula and software with funding from the National Science Foundation.

This content downloaded on Mon, 4 Mar 2013 13:40:01 PM
All use subject to JSTOR Terms and Conditions

Students using this method can compute joint probabilities without generating the entire tree diagram and for situations in which the elementary outcomes are not equally likely. However, they frequently do not understand why they are multiplying the probabilities.

One reason for students' confusion about the multiplication of probabilities is that, conceptually, probabilities are not multiplied. Instead, they are divided, or, to use a term Confrey (1990) prefers, they are "split." Tree diagrams tend to focus attention on the *multiplication* of possible *events* as subsequent levels of events are added rather than on the *splitting* of *probability* among these events. In fact, probabilities are not graphically depicted in tree diagrams; they appear only as numeric labels on the branches. The pipe diagram was devised as a way to represent graphically not only the collection of possible events—the sample space—but also the probabilities of those events.

THE PIPE DIAGRAM

Figure 2 is the pipe diagram for the same situation involving two coin flips. Two differences between it and the tree diagram are obvious. First, the branches, or pipes in this new metaphor, are tagged with *joint* rather than conditional probabilities. Second, the values of these joint probabilities are graphically depicted as pipe widths. As the name implies, the diagram is meant to suggest a system of pipes through which water flows. An event is represented by a particular pipe in the system, and the probability of an event is equivalent to the fraction of the total amount of water entering at the left that flows into a particular section of the pipe. Students are instructed in this example to think of P(HH) as the fraction of water that would flow into, or out of, the top pipe.

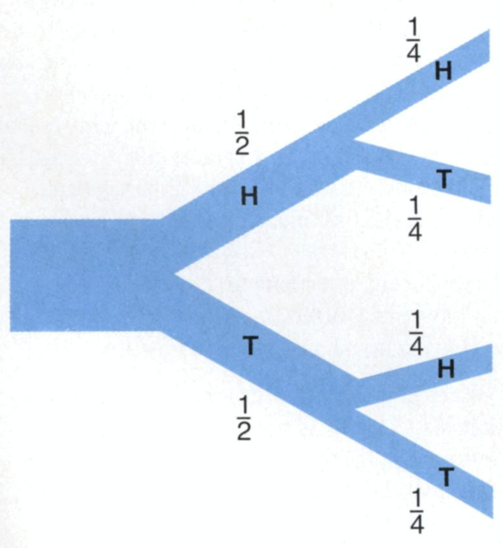


Fig. 2
Pipe diagram for two flips of a fair coin

In the pipe diagram, the joint probabilities in each column sum to 1, which is a concrete representation that the probabilities of all mutually exclusive outcomes of some random event sum to 1. This fact makes sense to students in terms of the pipe metaphor because all the water from one column or stack is split among the pipes in the next column and is therefore always equal to the original amount. By contrast, the sum of the accompanying conditional probabilities in all but the first column of the tree diagram is greater than 1.

Pipe diagrams give students a concrete reference

Using the tree diagram to solve the coin problem, students compute P(HH) by first drawing the entire diagram and then multiplying the conditional probabilities of each individual outcome. The pipe diagram requires the student to compute joint probabilities as each additional level is added. Experience suggests that although students may be slow to understand how multiplication by rational numbers gets the job done, they are quick to see, for example, that in computing P(HH), the probability of the prior, or parent, event, H, must be split; half goes to HT, leaving the other half to HH. Thus they can determine the relative size of the pipe associated with HH more quickly than they can determine the appropriate probability. Much of the instructive power of the pipe diagram comes from the fact that while constructing each part of it, the student is given a concrete reference not only for the value of the probability but also for the generating mathematical operation (splitting).

The differences between pipe and tree diagrams become more obvious with a problem in which the elementary events are not equally likely.

The basketball problem

> A basketball player is about to shoot two free throws that may win the game. Over the season, she has made 2/3 of her free throws. What is the probability that she makes both free throws?

To solve this problem given only this information, students must assume that the probability of making the second shot is not affected by the outcome of the first shot. Surprisingly, data obtained from professional players suggest that this assumption is true (Gilovich, Vallone, and Tversky 1985).

The pipe and tree diagrams for this problem are shown in **figure 3.** In the pipe diagram, the upper pipe of the first division is labeled H (Hit) and the lower pipe, M (Miss). The leftmost M pipe is half the width of the corresponding H pipe, reflecting the differences in probabilities between the two events. Two-thirds of the "water" flows through the first H pipe, and the rest, 1/3, flows through the first M pipe. At the second split, 2/3 of the water in

This content downloaded on Mon, 4 Mar 2013 13:40:01 PM
All use subject to JSTOR Terms and Conditions

The relative sizes of the joint probabilities are obvious

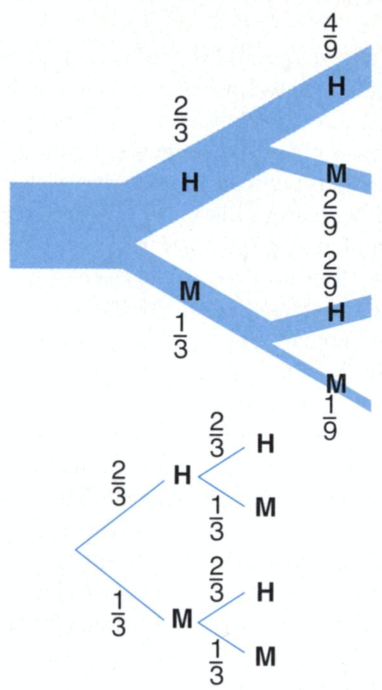


Fig. 3
Comparison of pipe and tree diagrams for the basketball problem

the first H pipe, or 4/9 of the original amount, continues on through a second column-H pipe. This pipe is associated with the probability of making both shots. The relative sizes of the various joint probabilities—HH is twice the thickness of HM—are obvious in the pipe diagram, whereas the tree diagram gives no graphical clue to these relations.

As the number of columns and elementary events increases, it becomes increasingly difficult to maintain the appropriate pipe widths. For this reason, students can be asked to determine probabilities of various complex events by extrapolating from a simpler pipe diagram. For example, imagine in a typical game that the player mentioned in the foregoing problem attempts ten free throws. What is the probability that she makes all ten? By examining the pipe diagram for two shots, students can see that for each additional shot, the pipe at the top will be reduced in size by a factor of 2/3. After a few iterations, many students can formalize the pattern using exponents to compute the probability of ten consecutive hits ($(2/3)^{10} \approx 0.017$). Getting students to perform these mental extensions should be one of the primary objectives when using either pipe or tree diagrams because neither diagram can represent problems with other than small sample spaces.

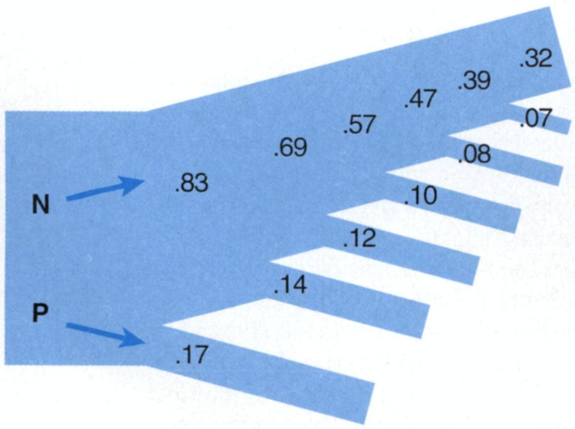


Fig. 4
Pipe diagram for the pregnancy problem (P = pregnant, N = not pregnant)

Another solution to dealing with multilevel sample spaces is to have students depict only the event of interest by "pruning" the diagram. They could compute the probability of making all ten shots by drawing only two pipes at each level, one corresponding to making the shot, the other, to missing. A pipe associated with a missed shot would not be split in the next level, since it is now part of a series of shots that includes at least one miss. Pruning is a more natural way to represent the "wait time" problem presented in the next section.

The pregnancy problem

The Johnsons have decided to have a baby. Unfortunately, they have jobs in different cities and are together only on Saturdays. They want to know about how long it will take them to conceive given the current situation. Through a little research, they discover that for fertile couples who have intercourse once a week, the probability that they conceive in a particular month is 0.17.

Only the upper "not pregnant" pipe in **figure 4** continues splitting because when pregnancy occurs, the "experiment" ends. The probabilities shown on the diagram are, again, joint probabilities. For example, the second column shows the joint probabilities of (1) not getting pregnant during either the first or second month (0.69) and (2) not getting pregnant in the first month and getting pregnant in the second (0.14).

The following questions can be asked after students have constructed this pipe diagram. The correct answers are shown in parentheses.

1. Use the diagram to determine the probabilities that Ms. Johnson is not pregnant after (a) the first month, (b) the second month, . . . , (f) the sixth month. (These probabilities are shown in the upper segment of the pipe diagram.)
2. What is the probability that Ms. Johnson is pregnant by the end of the third month? (0.17 + 0.14 + 0.12 = 0.43.)

This content downloaded on Mon, 4 Mar 2013 13:40:01 PM
All use subject to JSTOR Terms and Conditions

3. If Ms. Johnson is still not pregnant after the third month, what is the probability of her becoming pregnant in the fourth month? (0.17)
4. Without actually extending the pipe diagram, calculate the probability that Ms. Johnson is still not pregnant after one year. ($(0.83)^{12} \approx 0.11$.)

Many students will assume that the reason the probability of being pregnant after six months is so much larger than after one month, 0.68 compared with 0.17, is that the probability of getting pregnant increases with each failure. Question 3, however, stresses that given that Ms. Johnson has not yet conceived, the probability of getting pregnant in the coming month remains constant at 0.17. Visually, the lower pipes that split off from the top section are always of the same width relative to the upper section to the immediate left.

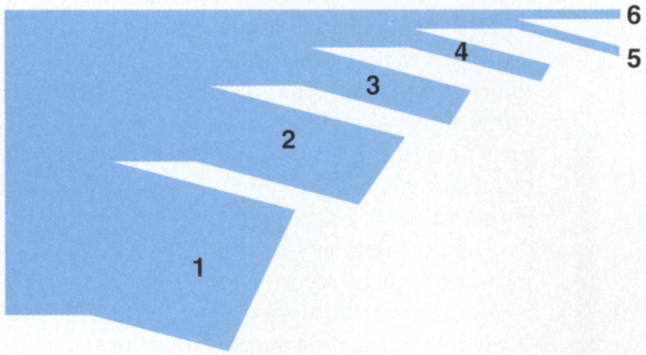
Question 4 asks students to imagine extending the diagram to determine a probability of an event that is not easily depicted graphically. In the process, some students learn to connect the physical representation of probability to a mathematical one, such as $(0.83)^{12}$.

I have used the pregnancy problem successfully on several occasions with older adolescents at SummerMath, a six-week residential mathematics program for high school–aged young women sponsored by Mount Holyoke College, and in a course taught by Al Gagnon at Holyoke High School in Massachusetts. I have never encountered an objection from either the students or their families. I present the problem matter-of-factly with a sense of confidence that students will discuss the problem maturely. They show considerably more interest in thinking about this situation than about the basketball problem, for example. Certainly, that the problem has for them obvious present and future implications and that the value of 17 percent is based on actual research (Barrett and Marshall 1969) have a lot to do with their motivation.

WATER-PIPE PROBLEMS

Before formally drawing an analogy between probability and water in a section of a pipe system, I give students problems similar to those shown in **figure 5.** Notice that the accompanying questions, with which students have little difficulty, are structurally comparable to typical probability computations. Again, questions are included to help students move from computations based on visual representation of the dividing process to formal techniques. Ideally, the derived expression $(1/2)^n$ in question 2c of **figure 5** becomes for the student not a static expression into which a value of n is plugged but a representation of the dynamic process of halving, which begins, in this example, at $n = 1$ and can continue indefinitely.

Six towns are served by the water system shown in the diagram. The pipe carries water from a large reservoir on the left to the six towns. Each time the pipe splits, it splits into two sections of equal width.



1. Determine the fraction of the water piped from the reservoir that goes to each town.
2. Future plans call for extending the pipe system to include several more towns being built farther down the pipe line.

a) If the system is divided in the same way that it has been to this point, what fraction of the total water coming from the reservoir will town 7 receive?

b) What fraction will go to town 12?

c) Let n stand for the number of a town. Note that a town's number tells you how many times the pipe has been divided before reaching that town. Write a mathematical expression, or rule, that would allow you to calculate the fraction of the water that town n would receive.

Fig. 5
Water-pipe problem

Readers may have noted that an actual water pipe half the diameter of another pipe would carry only one-fourth, not one-half, the volume of water. Pipe diagrams should therefore be regarded as two-dimensional pipes—plumbing in Flatland—or as cross sections of rectangular pipes. Perhaps unfortunately, I have never had a student point out the limits of the metaphor. Also, some costs are associated with using pipe diagrams: they require more space on the page than do tree diagrams, take about four times as long to draw, and require more time in early instruction to remind students to take the widths seriously. I tell my students that when they draw pipe diagrams, their pipe widths need remain only relatively accurate. Someone looking at the diagram should be able to tell that one event is more probable than another, but not necessarily how much more probable.

Students learn to connect physical and mathematical representations

INTERPRETING PROBABILITIES

In addition to helping students understand why and when probabilities are multiplied, pipe diagrams may foster a better understanding of what a probability is. Research suggests that many college students hold to a general view of probability described by Konold (1989) as an "outcome approach." One manifestation of this outcome approach is that

This content downloaded on Mon, 4 Mar 2013 13:40:01 PM
All use subject to JSTOR Terms and Conditions

when asked about the probability of some event, many students think that they are being asked to predict whether that event *will* occur rather than to quantify how likely, or often, it will occur. For example, these students interpret the forecast "70 percent chance of rain" to mean that "it will rain." When told that no rain fell on that forecasted day, these students reply that the forecast of 70 percent rain was incorrect. Probability values in the outcome approach are turned into definite predictions by judging whether they are sufficiently close to the anchor values of 100 percent ("yes") and 0 percent ("no"). In the weather example, 70 percent is regarded by many students as close enough to 100 percent to warrant the assertion that "it will rain." Given this frame of reference, it then makes sense why many students regard probabilities close to 50 percent as indications of ignorance, as synonymous with the statement "I don't know." For example, when asked to interpret the meaning of the forecast "50 percent chance of rain," one student in Konold's (1989) study replied in this manner:

Students often turn probability values into definite predictions

> I'd kind of think that was strange—that he didn't really know what he was talking about, because only 50-50—"it might rain or it might be sunny, I really don't know."

In a more recent study, Konold et al. (1993) found that the percent of undergraduate students holding this outcome-oriented view was unrelated to earlier instruction in statistics. This situation suggests that students can learn to compute various probabilities and still have a poor idea of what a probability is.

The way in which tree diagrams are typically described may inadvertently support this outcome-oriented reasoning. Teachers tend to talk about the nodes of the tree diagram as either-or decision points: "If the coin lands tails up on the first flip, what might happen on the second flip?" Unfortunately, this type of description fits well with some students' belief that their task is to predict which path will be taken rather than to determine probabilities associated with those paths. The pipe diagram may supply the basis for a more viable understanding by building on the analogy of water-filled pipes. Using the diagram is not a matter of deciding which way the water will flow, since it distributes itself throughout the system, but of figuring out how much is in various parts. Similarly, probability is not about predicting whether a particular event will occur but about determining how the probability is distributed over the possible events.

REFERENCES

Barrett, John C., and John Marshall. "The Risk of Conception on Different Days of the Menstrual Cycle." *Population Studies* 23 (1969):455–61.

Confrey, Jere. "Splitting, Similarity and Rate of Change: A New Approach to Multiplication and Exponential Functions." Paper presented at the annual meeting of the American Educational Research Association, Boston, 1990.

Gilovich, Thomas, Robert Vallone, and Amos Tversky. "The Hot Hand in Basketball: On the Misperception of Random Sequences." *Cognitive Psychology* 17 (July 1985):295–314.

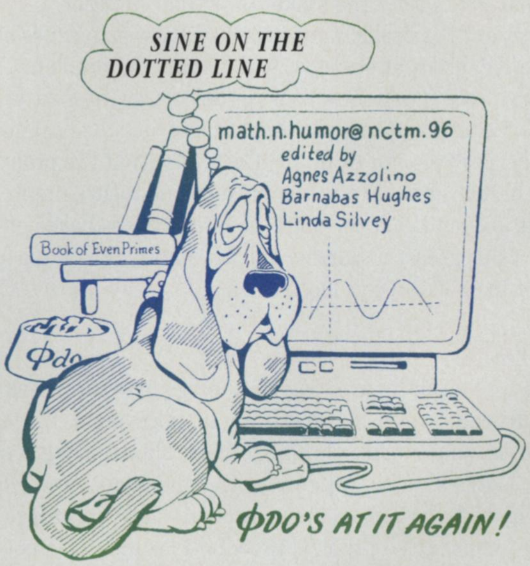
Kemeny, John G., Laurie J. Snell, and Gerald L. Thompson. *Introduction to Finite Mathematics.* Englewood Cliffs, N.J.: Prentice-Hall, 1956.

Konold, Clifford. "Informal Conceptions of Probability." *Cognition and Instruction* 6 (1989):59–98.

———. "Understanding Students' Beliefs about Probability." In *Radical Constructivism in Mathematics Education,* edited by Ernst von Glasersfeld, 139–213. Dordrecht, Holland: Kluwer, 1991.

Konold, Clifford, Alexander Pollatsek, Arnold Well, Jill Lohmeier, and Abigail Lipson. "Inconsistencies in Students' Reasoning about Probability." *Journal for Research in Mathematics Education* 24 (November 1993):392–414.

The writing of this article was funded by National Science Foundation grant no. MDR-8954626. The opinions expressed herein are the author's and are not necessarily those of the Foundation. The author thanks Amy Robinson, Ruma Falk, and Sandy Pollatsek for helpful comments on earlier drafts.



NCTM invites you to contribute humorous material for the revision of its popular Mathematics and Humor booklet. Send material to: Agnes Azzolino, Mathematical Concepts, Inc., 85 First St., Keyport, NJ 07735-1503.

This content downloaded on Mon, 4 Mar 2013 13:40:01 PM
All use subject to JSTOR Terms and Conditions