Dragging a Box: The Representation of Constraints and the Constraint of Representations

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In the November 2000 issue of *TPT*, W.H. van den Berg[1] shows us how students can analyze dragging a box with a rope and determine the "best" angle for starting the box moving by finding the angle that requires the smallest tension force. Near the conclusion of that note, van den Berg derives an expression for the best angle by differentiating the tension force with respect to the angle θ and doing quite a bit of algebra. I would like to show readers a derivation that does not require calculus or algebra. It requires only some geometry, some trigonometry, and some "lateral" thinking. Along the way, I will introduce readers to an alternative way of thinking about the normal and friction forces. Upon reflection, we will learn something about the possible limitations of the first viable representation we choose to solve a problem, the value of alternative representations, and how we as teachers typically represent the three "constraint" forces: normal, static friction, and tension.

A Calculus- and Algebra-free Derivation of the Best Angle to Drag a Box

The situation is shown in Fig. 1. There are four forces on the box: a gravitational force $F_g = Mg$ exerted by the Earth, a normal force F_N exerted by the floor, a static friction force F_{fs} also exerted by the floor, and a tension force F_T exerted by the rope.

We are looking for the smallest tension force needed to break static friction. As described in Ref. 1, the smallest force occurs when the person pulls at a slight angle (θ_{best}) above the horizontal. This reduces the normal and frictional forces sufficiently without reducing the horizontal component of the tension force so much as to make the other changes ineffective.

The first step in finding the best angle is to recognize that there are actually only *three* forces exerted on the box. What we conventionally refer to as the normal and frictional forces are really components of a single force: the force exerted by the floor. The normal force is perpendicular to the surface, and the frictional force is parallel to the surface.

The second step is to recognize that when the static friction force is maximal, the force exerted by the floor points in a particular direction. In other words, no matter how large or small the normal force is, the direction of the force exerted by the floor is always the same, because the maximum static friction force is proportional to the normal force (i.e., $F_{fs,max} = \mu_s F_N$). When one component changes, the other changes by the same proportion.

The third step is to draw a "vector-addition diagram" for the box rather than a free-body diagram. This is done by drawing the forces head-to-tail and results in a vector representation of the net force. In this situation, because the acceleration of the box is zero, the net force is zero.

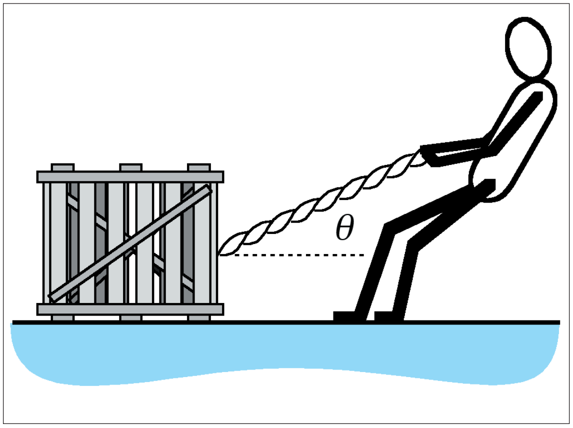

Fig. 1. Dragging a box.

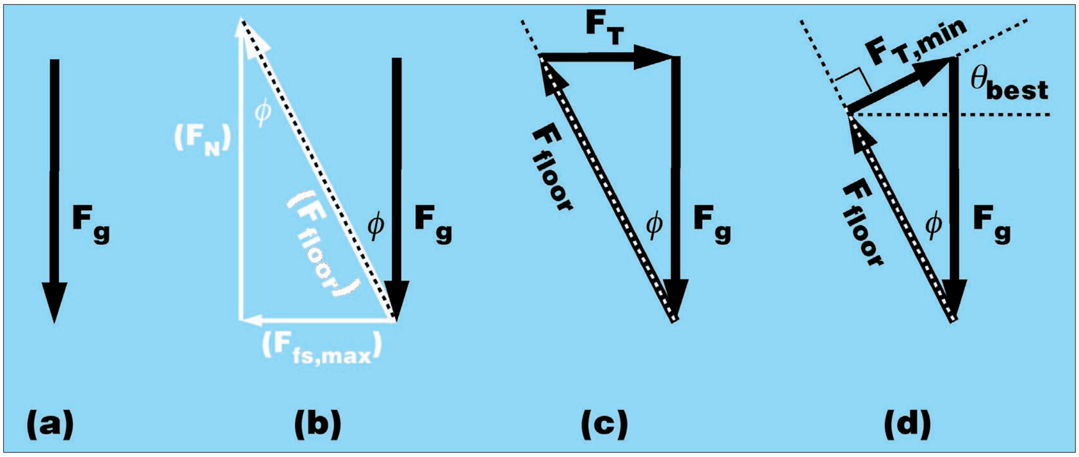

Fig. 2. Creating a vector-addition diagram. (a) Gravitational force is unchanging. (b) When static friction force is maximal, force exerted by floor is in same direction regardless of F_T. (c) Tension force when pulling horizontally. (d) Finding smallest tension force.

And because there are three forces on the box, the vector-addition diagram is a triangle.

Of the three forces on the box, only the gravitational force is unchanging, so we start by drawing it [see Fig. 2(a)]. The *strength* of the force exerted by the floor is changing, but its direction is not, so we represent it (for now) as a dotted line [see Fig. 2(b)]. The angle this line makes with the vertical is $\phi = \tan^{-1}(F_{fs,max}/F_N) = \tan^{-1}\mu_s$.

The third and final force is the tension force. When the person pulls horizontally, for example, the tension force is represented by a horizontal arrow beginning at the dotted line and ending at the tail of the arrow that represents the gravitational force [Fig. 2(c)]. The result is $F_T = F_g \tan\phi = \mu_s Mg$, as expected.

To find the *smallest* force, we must ask ourselves: What is the smallest arrow that can be drawn beginning somewhere along the dotted line and ending at the tail of the gravitational force? The answer is: The arrow that is perpendicular to the dotted line. This is shown in Fig. 2(d). The angle that this tension force makes with the horizontal is ϕ. Therefore, $\theta_{best} = \phi = \tan^{-1}\mu_s$. This is the same result that is usually found using calculus and a considerable amount of algebra.

We can also write down an expression for the smallest tension force. Again using Fig. 2(d), we get $F_{T,min} = F_g \sin\phi = \mu_s Mg/(1 + \mu_s^2)^{1/2}$. No calculus and little algebra are needed for this result.

Other Examples

The simplest extension of these ideas is to the analysis of a moving box. The frictional force is $F_{fk} = \mu_k F_N$, so again, the force exerted by the floor has the same direction (relative to the floor) no matter what the situation is, as long as the box is sliding. The best angle to drag the box is now $\theta_{best} = \tan^{-1}\mu_k$, and the smallest tension force is now $F_{T,min} = \mu_k Mg/(1 + \mu_k^2)^{1/2}$.

Another example is shown in Fig. 3. A block is at rest on a rough wedge that is on a rough horizontal surface. The question here is: What is the frictional force exerted by the horizontal surface on the wedge?

The UMass Physics Education Research Group has posed this question many times, to both teachers and students, and the results highlight the importance of the choice of representation while reasoning. A surprising number of people — even experienced physics teachers — have difficulty with this situation because they usually start thinking about the interactions between the block and the wedge, which leads them to draw or imagine a conventional free-body diagram for the block (not the wedge), as

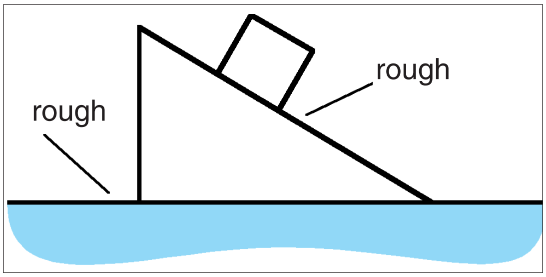

Fig. 3. Block at rest on a wedge.

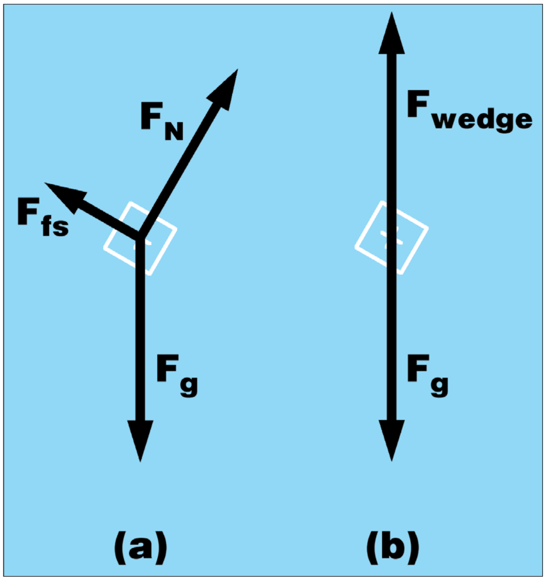

Fig. 4. Free-body diagrams for block (a) in terms of components of force exerted by wedge, and (b) not in terms of these components.

shown in Fig. 4(a).

In this representation, there are three forces on the block: a gravitational force F_g exerted by the Earth, a normal force F_N exerted by the wedge, and a static friction force F_{fs} also exerted by the wedge. In part because free-body diagrams are seldom drawn perfectly to scale, and in part because it is difficult to reason about the horizontal components of these forces without deeper analysis, students and teachers often convince themselves that there is a net horizontal force exerted by the block that must be balanced by a frictional force on the wedge due to the horizontal surface. (It is uncommon for people to realize that the horizontal components of F_N and F_{fs} must balance in order for the block to remain at rest.)

Alternatively, if we think of the normal and static friction forces as components of a single force exerted by the wedge on the block, the analysis becomes trivial. In this representation, there are only two forces on the block, and since the block is at rest, these forces are equal and opposite, as shown in the free-body diagram in Fig. 4(b).

Avoiding any confusion about the frictional force exerted by the horizontal surface, we see that only a vertical force is needed to keep the wedge at rest, because there is no horizontal component to the force exerted on the wedge by the block. Therefore, no frictional force is exerted by the surface on the wedge.

Comments

Many of us don't change representations once we've found a viable one, and the choice of representation can often dictate how a problem is solved. As we've seen when looking for the "best" angle to drag a box, if we use a free-body diagram and treat the normal and frictional forces independently, the only option is to use calculus. Yet the choice of representation can mean the difference between making a problem easy or difficult to solve, and students should be constantly making choices about *how* they will solve a problem before attempting a solution. As teachers, we can become overly programmed when thinking about concepts such as force, causing students to become even more constrained in how they analyze and reason about situations.

Alternative representations can bring out limitations in how we conceptualize problem situations. They also give students a second approach, which can improve their efficiency and permit self-checking. Finally, alternative representations can enrich instruction by providing additional links between ideas as well as opportunities to go beyond the conventional.

Reference

1. Willem H. van den Berg, "The best angle for dragging a box," *Phys. Teach*. **38**, 506 (Nov. 2000).